# Some vanishing and finiteness results on complete manifolds: a generalization of the Bochner technique

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In this talk we present some results, recently obtained in collaboration with M. Rigoli and A.G. Setti, that extend the original Bochner technique to the case of $L^p$ harmonic forms on geodesically complete manifolds and in the presence of an amount of negative curvature.

## Basic references.

[1] S. P., M. Rigoli, A.G. Setti, *Vanishing theorems on Riemannian manifolds and applications.* J. Funct. Anal. **229** (2005), 424–461.

[2] S. P., M. Rigoli, A.G. Setti, *A finiteness theorem for the space of $L^p$ harmonic sections.* To appear in Rev. Mat. Iberoamer.

[3] S. P., M. Rigoli, A.G. Setti, *Topics in geometric analysis: vanishing and finiteness results on complete manifolds.* Book in preparation.

We will move in the realm of **Geometric Analysis**.

Roughly speaking: you are given a geometric problem. Summarize it into a family of functions (of geometric content) which, in turn, are governed by a system of differential (in)equalities. Obtain information on the qualitative and quantitative properties of solutions of these differential systems. Geometry, in general, will impose some further constrains and guide the analysis of solutions. Apply this information to the given geometric functions and get a conclusion about the original problem.

A prototypical example: the celebrated **Bochner technique,** originally introduced by S. Bochner in the '50s to investigate the relation between the topology and the curvature of a closed (i.e. compact and without boundary) Riemannian manifold.

# The case of a closed manifold

## Bochner original argument

Original question: may one prescribe the sign of the curvature on a generic smooth, closed manifold?

Let $M$ be a smooth, compact manifold. Then, there is a contractible, open set $\mathcal{E} \subseteq M$, with $\overline{\mathcal{E}} = M$, such that $\mathcal{E}$ supports a metric with constant curvature of a prescribed sign. Simply fix any metric $(,)$ on $M$, a reference origin $p \in M$ and delete from $M$ the corresponding cut-locus $cut\,(p)$, which is a closed (hence compact) set of zero-measure. Thus $\mathcal{E} = M - cut\,(p)$ is diffeomorphic to the star-shaped, relatively compact, open set $0 \in E \subseteq T_pM \approx \mathbb{R}^m$ via the exponential map $\exp_p$. To conclude, fix a constant curvature metric on $E$ and pull it back on $\mathcal{E}$.

**Remark.** In a (quite strong) sense, the topology of $M$ is contained in the (apparently evanescent) removed set $cut\,(p)$, e.g., the inclusion $i : cut\,(p) \hookrightarrow M$ induces isomorphisms between homology (and cohomology) groups

$$H_k\,(cut\,(p)\,;\mathbb{Z}) \simeq H_k\,(M;\mathbb{Z})$$

at least for $k \neq m, m-1$.

Now, closing $M - cut\,(p)$ by addition of $cut\,(p)$ produces a non-trivial topology that, in general, may represent an obstruction for $M$ to support a Riemannian metric with some curvature bound, e.g., given sign. Bochner result goes precisely in this direction. Let us recall the argument.

**Theorem 1 (Bochner)** *Let* $(M, \langle , \rangle)$ *be a connected, closed, oriented, Riemannian manifold,* $m = \mathrm{dim}\, M$*. Set* $b^1(M; \mathbb{R})$ *for the first (real) Betti number of* $M$*. Then*

$$Ric \geq 0 \text{ on } M \Longrightarrow b^1(M; \mathbb{R}) \leq m$$

*the equality holding if and only if* $M$ *is a flat torus. Furthermore,*

$$Ric > 0 \text{ at some } p \in M \Longrightarrow b^1(M; \mathbb{R}) = 0.$$

**Proof.** (From Geometry to Analysis) Define the Hodge-Laplacian as

$$\Delta_H \omega = (d\delta + \delta d)\, \omega = 0.$$

where $d$ is the exterior differential and $\delta$ stands for the (formal) adjoint of $d$ with respect to the $L^2$ inner product of $k$-forms. Set

$$\mathcal{H}^k(M) = \{k\text{-forms } \omega : \Delta_H \omega = 0\},$$

the vector space of harmonic $k$-forms on $M$. By Hodge-de Rham theory

$$b^1(M; \mathbb{R}) = \dim \mathcal{H}^1(M).$$

Weitzenbock-Bochner formula states that, for $\omega \in \mathcal{H}^1(M)$,

$$\text{(BW)} \qquad \frac{1}{2}\Delta |\omega|^2 = |D\omega|^2 + Ric\left(\omega^\#, \omega^\#\right),$$

where $\Delta$ is the Laplace-Beltrami operator ($+d^2/dx^2$ on $\mathbb{R}$) and $D$ denotes the extension to 1-forms of the Levi-Civita connection of $M$. Suppose $Ric \geq 0$. By assumptions and (BW)

$$\Delta |\omega|^2 \geq 0, \text{ i.e,. } |\omega|^2 \text{ subharmonic}$$

Note that: $M$ closed implies $|\omega| =$const. Two different viewpoints:

(a) $L^\infty$ viewpoint. The smooth function $|\omega|$ attains its maximum at some point and, therefore, by the **Hopf maximum principle** we conclude that $|\omega| =$const.

(b) $L^p$ viewpoint. Use the **divergence theorem**:

$$0 = \int_M \operatorname{div}\left(|\omega|^2 \nabla |\omega|^2\right) = \int_M \left|\nabla |\omega|^2\right|^2 + |\omega|^2 \Delta |\omega|^2 \geq \int_M \left|\nabla |\omega|^2\right|^2 \geq 0.$$

This again implies $|\omega| =$const.

Use this information into (BW) formula:

$$\begin{aligned}(BW) &\Longrightarrow D\omega = 0\\ &\Longrightarrow \omega \text{ is determined by its value at any } p \in M.\end{aligned}$$

Fix $p$. The evaluation map $\varepsilon_p(\omega) = \omega_p : \mathcal{H}^1(M) \to \Lambda^1\left(T_p^* M\right)$ is an injective homomorphism. Therefore

$$\dim \mathcal{H}^1(M) \leq m.$$

Note that

$$(BW) \Longrightarrow Ric\left(\omega_p^{\#}, \omega_p^{\#}\right) = 0, \text{ at } p.$$

Therefore,

$$Ric(p) > 0 \Longrightarrow \omega_p = 0 \Longrightarrow \omega \equiv 0 \Longrightarrow \dim \mathcal{H}^1(M) = 0.$$

■

**Remark.** Crucial fact in the above proof:

$$Ric \geq 0 \Longrightarrow \Delta |\omega| \geq 0.$$

**Question.** What happens in the presence of an amount of negative curvature?

**Answer.** In general, there is no uniform bound of $\dim \mathcal{H}^1(M)$, i.e., no uniform control on the topology.

**Example.** Let $S$ be an orientable, closed Riemann surface of genus $g \geq 2$, by uniformization (and recalling the Gauss-Bonnet theorem) we can endow $S$ with a Riemannian metric of Gauss curvature $-1$.

**Analytical counterpart.** Set

$$-R(x) = \min_{v \in \mathbb{S}^{m-1} \subset T_x M} Ric_x(v, v),$$

the pointwise lower bound of the Ricci tensor. From (BW) we have the **Bochner inequality**

$$(*) \qquad \frac{1}{2}\Delta |\omega|^2 + R(x)|\omega|^2 \geq |D\omega|^2 \geq 0.$$

We would like to get $LHS(*) = 0$. But, in general, the maximum principle fails to hold for inequalities of this type. Divergence theorem does not help us.

**Remark.** A fundamental result by M. Gromov [Comm. Math. Helv. 1981] states that a uniform limitation on the Betti numbers of a close manifold is obtained by requiring a control on a further Riemannian invariant, namely, the diameter. From a different (more analytic) perspective, we shall see momentarily how one could think of extending Bochner estimating theorem in the presence of (a little amount of) negative curvature.

# Generalized maximum principle

We are given a solution $\psi \geq 0$ of

(*) $$\Delta\psi + q(x)\psi \geq 0.$$

**Main assumption.** Assume $M$ supports a function $\varphi > 0$ such that

$$\Delta\varphi + q(x)\varphi \leq 0.$$

**Remark.** The existence of $\varphi$ is related to spectral properties of the operator $-\Delta - q(x)$.

**Idea.** To absorb the linear term of (*) using a combination of the solutions $\psi$ and $\varphi$.

Set

$$u = \frac{\psi}{\varphi} \Longrightarrow \Delta u + \langle \nabla u, \nabla \log \varphi \rangle \geq 0.$$

There is no linear (i.e. zero-order) term in $u$. Therefore, the Hopf maximum principle applies.

$$M \text{ closed} \Longrightarrow u \text{ attains maximum} \overset{\text{Hopf}}{\Longrightarrow} u \equiv const.$$

Equivalently,

$$\psi = C\varphi, \ C \geq 0.$$

Use the differential inequalities satisfied by $\psi$ and $\varphi$ and deduce that, in fact,

$$\Delta\psi + q(x)\psi = 0.$$

Special case:

$$\omega \in \mathcal{H}^1(M), \ \psi = |\omega|^2, \ q(x) = 2R(x),$$

with $R(x)$ the lower Ricci bound. Bochner inequality yields

$$0 = \frac{1}{2}\Delta |\omega|^2 + R(x)|\omega|^2 \geq |D\omega|^2 \geq 0.$$

Once again, $\omega$ is parallel, thus extending the original Bochner result.

**Remark.** Let $M$ be closed (parabolic suffices).

If $R(x) \geq 0$, then

$$\Delta\varphi \leq -2R(x)\varphi \leq 0.$$

The superharmonic function $\varphi > 0$ must be contstant. Hence $R(x) \equiv 0$. As a consequence (in the compact - parabolic - setting) the Main assuption represents a genuine extension of Bochner condition $Ric \geq 0$ only in case $R(x)$ changes its sign.

# The setting of open manifolds

**Question.** What does of the previous picture survive in the case of a non-compact manifold $(M, \langle , \rangle)$?

Examples help us to understand the situation. We shall consider the general case of harmonic $k$-forms, any $k$. First, we need to introduce some more notations and inequalities.

# Bochner(-type) and Kato inequalities

Let $(M, \langle,\rangle)$ be any manifold, $m = \mathbf{dim}\, M$. We consider $k$-forms, any $k$. Assume:

(a) case $k = 1$, $\qquad Ric \geq -R(x)$.

(b) case $k > 1$, $\qquad \rho_x \geq -R(x)$,

where $\rho_x : \Lambda^2(T_x M) \to \Lambda^2(T_x M)$ is the curvature operator.

Take $\omega \in \mathcal{H}^k(M)$. Then, by Gallot-Meyer [J. Math. pures et appl. 1973], the following **Bochner inequality** holds

$$\frac{1}{2}\Delta|\omega|^2 + CR(x)|\omega|^2 \geq |D\omega|^2 \geq 0,$$

for a suitable $C = C(k, m) > 0$. E.g. $C = k(m-k)$ if $M = \mathbb{H}^m_{-1}$.

Direct computations show that

$$|\omega| \{\Delta |\omega| + CR(x) |\omega|\} \geq |D\omega|^2 - |\nabla |\omega||^2,$$

The sign of the RHS: in general, one has the **Kato inequality**

$$|D\omega|^2 - |\nabla |\omega||^2 \geq 0.$$

In case $\omega$ is both closed and co-closed, i.e.,

$$d\omega = 0, \quad \delta\omega = 0,$$

then we have the **refined Kato inequality**

$$|D\omega|^2 - |\nabla |\omega||^2 \geq A |\nabla |\omega||^2,$$

for a suitable constant $A = A(m, k) > 0$. E.g. $k = 1 \Longrightarrow A = 1/(m-1)$

**Notation.**

$$L^p\mathcal{H}^k(M) = \left\{\omega \in \mathcal{H}^k(M) : |\omega| \in L^p\right\}.$$

**Remarks.**

1. Alexandru-Rugina [Rend. Sem. Mat. Univ. Politec. Torino 1996]

$$\omega \in L^{p\neq 2}\mathcal{H}^k(M) \not\Longrightarrow d\omega = 0 \text{ nor } \delta\omega = 0.$$

2. Gaffney [Annals 1954] Global integration by parts:

$$\left.\begin{array}{c} \omega \in L^2\mathcal{H}^k(M) \\ + \\ (M, \langle , \rangle) \text{ complete} \end{array}\right\} \Longrightarrow d\omega = 0,\ \delta\omega = 0$$

$$\Longrightarrow \text{Refined Kato.}$$

Conclusion: take $\omega \in \mathcal{H}^k(M)$. Then $\psi = |\omega| \geq 0$ satisfies an inequality of the form

(*) $$\psi \{\Delta\psi + q(x)\psi\} \geq A |\nabla\psi|^2,$$

with

$$q(x) \in C^0, \text{ and } A \in \mathbb{R}.$$

We shall refer to (*) as the **general Bochner-type inequality**.

# General existence result

Bochner result is essentially a vanishing&estimating theorem for the space of harmonic forms. Harmonic forms on a closed manifold represent cohomology classes: therefore their vanishing or their presence is a topological question. In contrast, in the non-compact setting, harmonic forms may represent nothing, even for a geodesically complete manifold $(M, \langle , \rangle)$.

**Example.** On the flat Euclidean space $\mathbb{R}^m$ every differential $k$-form $h(x)\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ is harmonic provided $h(x)$ is a harmonic function. Now, the space of harmonic functions on $\mathbb{R}^m$ is not finitely generated (e.g. harmonic polynomials). Therefore

$$\dim \mathcal{H}^k(\mathbb{R}^m) = +\infty.$$

**Remark.** Let $(M, \langle , \rangle)$ be a **generic open manifold**. Then, for every $k$, $\dim \mathcal{H}^k(M) \neq 0$.

Fix $\Omega \subset\subset M$, $\partial\Omega \in C^{\infty}$. Fix $\omega_0 \in \Lambda^k\left(\overline{\Omega}\right)$ with non-zero tangential (or normal) part on $\partial\Omega$.

Duff and Spencer [Annals 1952]$\Longrightarrow \exists ! \omega \neq 0$ solution of

$$\begin{cases} \Delta_H \omega = 0, & \text{on } \Omega \\ \omega = \omega_0, & \text{on } \partial\Omega. \end{cases}$$

Choose $D \subset\subset \Omega$ so that $M - D$ has no compact components.

$$\Delta_H \omega = 0 \overset{\text{uniquer continuation}}{\Longrightarrow} \omega \not\equiv 0 \text{ on } D.$$

Malgrange [Ann. Inst. Fourier 1955-1956]$\Longrightarrow$we can uniformly approximate $\omega$ on $D$ by a harmonic $k$-form on $M$, say $\eta \in \mathcal{H}^k(M)$.

Take $\eta$ sufficiently close to $\omega$ on $D \Longrightarrow \eta \not\equiv 0$. Thus $\eta \in \mathcal{H}^k(M) \neq 0$.

# The need of some integrability condition

**Analytic counterpart of the above non-vanishing**. Suppose the curvature operator $\rho$ of $(M, \langle\, ,\rangle)$ satisfies $\rho \geq -R(x)$. Then, a harmonic $k$-form $\omega \in \mathcal{H}^k(M)$ satisfies

$$\frac{1}{2}\Delta|\omega|^2 + CR(x)|\omega|^2 \geq |D\omega|^2 \geq 0,$$

with $C = C(m,k) > 0$. Restrict now to $\rho \geq 0$, so that $R(x) \equiv 0$ and the above reduces to

$$\Delta|\omega|^2 \geq 0.$$

Following Bochner original argument, one tries to reach the Liouville-type conclusion $|\omega| = \mathbf{const}$ and, hence, that $\omega$ is parallel. However, in general, $|\omega|$ is neither (i) $L^\infty$, nor (ii) in some $L^{p<+\infty}$ integrability class and we have no Liouville property at all.

It happens that the geometry of the underlying manifold $(M, \langle , \rangle)$ enters the game when we consider harmonic functions and forms with these special integrability properties. The $L^\infty$ and $L^{p<+\infty}$ situations are substantially different. They are dealt with completely different methods. The $L^\infty$ case is often considered in the perspective of the weak maximum principle at infinity. We shall not consider this situation here. For an extensive study of the subject, we refer to a paper by P.-Rigoli-Setti, [Memoirs AMS 2005].

**In this talk we will focus our attention on the $L^{p<\infty}$ case.**

**Example.** The case of $\mathbb{R}^m$ is easy to handle and, therefore, can be used to exemplify the situation. Let $L^p\mathcal{H}^k(\mathbb{R}^m)$ be the vector space of the harmonic $k$-forms $\omega$ on $\mathbb{R}^m$ satisfying $|\omega| \in L^p(\mathbb{R}^m)$. We show that

$$\dim L^p\mathcal{H}^k(\mathbb{R}^m) = 0.$$

To see this, take any $\omega \in L^p\mathcal{H}^k(\mathbb{R}^m)$. Then, by Bochner-Weitzenbock, $|\omega| \in L^p(\mathbb{R}^m)$ is a subharmonic function. Since non-negative subharmonic functions in $\mathbb{R}^m$ enjoy the $L^p$ mean-value property we get, for any fixed $x_0 \in \mathbb{R}^m$ and for every $R > 0$,

$$|\omega|^p(x_0) \leq \frac{\int_{B_R(x_0)} |\omega|^p}{vol(B_R(x_0))}.$$

Letting $R \to +\infty$ we deduce $|\omega|(x_0) = 0$. Since $x_0$ was an arbitrary point, we conclude $\omega \equiv 0$, as claimed. The same proof works for a geodesically complete manifold $(M, \langle , \rangle)$ satisfying $\rho \geq 0$.

# Controlling the negative part of the curvature

**Example.** Let $\mathbb{H}^m$ be the standard hyperbolic space of dimension $m$ and constant curvature $-1$. We realize it as the Poincarè disc

$$\left(\mathbb{B}_1^m(0), \frac{4\sum dx^i \otimes dx^i}{\left(1-|x|^2\right)^2}\right),$$

where $\mathbb{B}_1^m(0)$ is the Euclidean unit ball of $\mathbb{R}^m$. Note that the hyperbolic metric is a pointwise conformal deformation of the Euclidean one. Note that, in general, if $\widetilde{\langle,\rangle} = \lambda^2(x)\langle,\rangle$ then

$$L^2\mathcal{H}^k(M,\langle,\rangle) \simeq L^2\mathcal{H}^k\left(M,\widetilde{\langle,\rangle}\right)$$

provided

$$2k = m = \dim M.$$

Specifying the situation to the Hyperbolic space we deduce

(*) $$L^2\mathcal{H}^k(\mathbb{H}^m) \simeq L^2\mathcal{H}^k(\mathbb{B}_1^m(0)).$$

Now, we have already observed that $\dim \mathcal{H}^k(\mathbb{R}^m) = +\infty$.

Since $\mathbb{B}_1^m$ has finite Euclidean volume, every $\omega \in \mathcal{H}^k(\mathbb{R}^m)$ restricts to a form $\omega' \in L^2\mathcal{H}^k\left(\mathbb{B}_1^m(0)\right)$.

By unique continuation, the restriction map is injective.

It follows that $\dim L^2\mathcal{H}^k\left(\mathbb{B}_1^m(0)\right) = +\infty$. According to (*), we conclude

$$\dim L^2\mathcal{H}^k\left(\mathbb{H}^{2k}\right) = +\infty.$$

# Bottom of the spectrum and Morse index

We introduce a "measure" of the negative part of the curvature in such a way that, "small" negative curvature implies finite dimensionality and, furthermore, small enough gives vanishing. This will be done using spectral properties of a suitable Schrodinger operator. From now on, $(M, \langle , \rangle)$ will denote a geodesically complete, non-compact, connected Riemannian manifold of dimension $m = \dim M$.

Suppose the curvature operator of $M$ satisfies $\rho \geq -R(x)$, $R(x) \geq 0$, so that, for every $\omega \in \mathcal{H}^k(M)$, its length $\psi = |\omega|$ satisfies

$$\psi \{\Delta\psi + CR(x)\psi\} \geq A |\nabla\psi|^2,$$

where $C = C(k, m) > 0$ and $A = A(k, m) \geq 0$, with $A > 0$ if $\omega$ is both closed and co-closed. Let us consider the Schrodinger operator

$$\mathcal{L} = -\Delta - q(x), \; q(x) = CR(x)$$

For any smooth $\Omega \subset\subset M$, the operator $(\mathcal{L}_\Omega, C_c^\infty(\Omega))$ has discrete spectrum

$$\lambda_1(\mathcal{L}_\Omega) < \lambda_2(\mathcal{L}_\Omega) \leq \lambda_3(\mathcal{L}_\Omega) \leq \ldots \leq \lambda_{N_0}(\mathcal{L}_\Omega) < 0 \leq \lambda_{N_0+1}(\mathcal{L}_\Omega) \leq \ldots$$

**Definition.** $\quad Ind(\mathcal{L}_\Omega) = N_0 = \#\text{negative eigenvalues.}$

By domain monotonicity of eigenvalues

$$\Omega_1 \subset \Omega_2 \Longrightarrow \lambda_k\left(\mathcal{L}_{\Omega_2}\right) \leq \lambda_k\left(\mathcal{L}_{\Omega_1}\right) \Longrightarrow Ind\left(\mathcal{L}_{\Omega_1}\right) \leq Ind\left(\mathcal{L}_{\Omega_2}\right),$$

**Definition.** $\quad Ind(\mathcal{L}_M) = \sup_{\Omega \nearrow M} Ind(\mathcal{L}_\Omega) \leq +\infty.$

**Definition.**

$$\lambda_1(\mathcal{L}_M) = \inf_{\Omega \nearrow M} \lambda_1(\mathcal{L}_\Omega) = \inf_{\varphi \in Lip_c(M)} \frac{\int |\nabla \varphi|^2 - \int q(x)\varphi^2}{\int \varphi^2} \geq -\infty.$$

Finiteness and vanishing of the Morse index essentially reflect the fact that the (positive part of the) potential is small in some integral sense.

**Theorem 2 (G.V. Rosenbljum, W. Cwikel, E. Lieb)** *Let $(M^m, \langle,\rangle)$ be complete and support the $L^2$-Sobolev inequality*

$$S_\mu \left(\int_M v^{2\mu}\right)^{\frac{1}{\mu}} \leq \int_M |\nabla v|^2 \, , \, \forall v \in C_0^\infty(M)$$

*for some $\mu > 1$ and some constant $S_\mu > 0$. Let $\mathcal{L} = -\Delta - q(x)$ where*

$$q_+(x) = \max(q(x), 0) \in L^{\frac{\mu}{\mu-1}}(M),$$

*Then, $\mathcal{L}$ is a semi-bounded, essentially self-adjoint operator on $L^2(M)$ with non-negative essential spectrum. Let $N_0$ be the number (counting multiplicity) of strictly negative eigenvalues of $\mathcal{L}$. Then, $\exists C = C(m, S_\mu) > 0$ such that*

$$N_0 = Ind(\mathcal{L}_M) \leq C \|q_+\|_{L^{\frac{\mu}{\mu-1}}(M)}.$$

# PDE counterparts of the spectral properties

**Theorem 3** *Let* $\mathcal{L} = -\Delta - q(x)$*. Then*

*(a) R. Gulliver 1984, Fisher Colbrie [Invent. 1985]. For any domain* $W \subseteq M$*,*

$$\lambda_1(\mathcal{L}_W) \geq 0 \Longleftrightarrow \exists \varphi > 0 \text{ solution of } -\mathcal{L}\varphi \leq 0 \text{ on } W.$$

*(b) Moss-Piepenbrink [Pacific Math. J. 1978], Fisher Colbrie-Schoen [C.P.A.M. 1980].*

$$Ind(\mathcal{L}_M) < +\infty \Longrightarrow \exists K \subset\subset M \text{ such that } \lambda_1(\mathcal{L}_{M-K}) \geq 0$$

*In particular*

$$Ind(\mathcal{L}_M) < +\infty \Longrightarrow \exists \varphi > 0 : -\mathcal{L}\varphi \leq 0 \text{ on } M - K, \text{ for some } K \subset\subset M.$$

**Remark.** The bottom-of-the-spectrum condition on $M-K$ is slightly weaker than the finiteness of the Morse index. Moreover, it is easier to handle. For instance, suppose $(M,\langle,\rangle)$ enjoys a global $L^2$-Sobolev inequality

$$\left(\int |u|^{2\mu}\right)^{\frac{1}{\mu}} \leq S_\mu^{-1} \int |\nabla u|^2 \,,\ \forall u \in C_c^\infty(M)$$

with $\mu > 1$, $S_\mu > 0$. Then, direct application of Hölder inequality gives

$$\|q\|_{L^{\frac{\mu}{\mu-1}}(M)} < +\infty \Longrightarrow \lambda_1\left(\mathcal{L}_{M-K}\right) \geq 0,$$

for a sufficiently large $K \subset\subset M$. Furthermore

$$\|q\|_{L^{\frac{\mu}{\mu-1}}(M)} \leq S_\mu \Longrightarrow \lambda_1\left(\mathcal{L}_M\right) \geq 0.$$

**Remark.** The PDE reformulation of the spectral properties is very suitable for an analytic approach to vanishing and finiteness results in the spirit of the generalized maximum principle.

# Main theorems: Bochner generalized

## Vanishing

Vanishing results for $L^2$ harmonic sections on complete manifolds under spectral assumptions go back to a paper by W. Elworthy and S. Rosenberg, [Acta Appli. Math. 1988]. Their technique relies on very refined probabilistic tools and requires the additional curvature assumption $\inf_M Ric > -\infty$ in order to guarantee that Brownian paths do not explode (a.s.) in a finite time. Soon later, P. Berard, [Manuscripta Math. 1990], generalized Elworthy-Rosenberg results by removing the curvature condition. Moreover, his proof is completely elementary and makes a direct use of the spectral assumption. He gets conclusion only in case of $L^2$ energies.

**Theorem 4 (P.-Rigoli-Setti, J.F.A. 2005)** *Let* $(M, \langle\ ,\ \rangle)$ *be a complete manifold,* $a(x) \in L^{\infty}_{loc}(M)$ *and let* $0 \leq \psi \in Lip_{loc}(M)$ *satisfy the differential inequality*

(*) $$\psi\Delta\psi + a(x)\psi^2 \geq A|\nabla\psi|^2 \qquad \text{weakly on } M.$$

*for some* $A \in \mathbb{R}$*. Suppose that there exists* $\varphi \in Lip_{loc}(M)$ *satisfying*

$$\Delta\varphi + Ha(x)\varphi \leq 0 \text{ weakly on } M$$

*for some* $H$ *such that*

$$H \geq -A + 1,\ H > 0$$

*If*

$$\psi \in L^{2p}(M)$$

*for some*

$$-A + 1 \leq p \leq H,\ p > 0,$$

*then there exist a constant* $C \geq 0$ *such that*

$$C\varphi = \psi^H.$$

*Further,*

*(i) If* $H - 1 > -A$*, then* $\psi$ *is constant on* $M$*, and if in addition,* $a(x)$ *does not vanish identically, then* $\psi$ *is identically zero;*

*(ii) If* $H - 1 = -A$*, and* $\psi$ *does not vanish identically, then* $\varphi$ *and therefore* $\psi^H$ *satisfy (*) with equality sign.*

## Finiteness

In case of $L^2$ harmonic sections, finiteness results have been extensively investigated by many authors under different assumptions. We limit ourselves to quote the “minimal hypersurfaces" papers [J. reine angew. Math. 2004; Math. Res. Let. 2002] by P. Li and J. Wang, where Morse index assumptions are used in a way similar to the present note, and the “$L^2$-cohomology paper"[Math. Ann. 1999], by G. Carron where quantitative dimensional estimates are obtained assuming that the underlying manifold supports a global Sobolev inequality; see also [Duke 1998, G.A.F.A. 2003].

**Remark**. The classical minimal (hyper)surface theory in Euclidean space greatly influenced the investigation of finiteness results on non-compact spaces under spectral properties of the relevant Schrodinger operator (the stability operator $\mathcal{L} = -\Delta - |II|^2$). According to the harmonic function theory by P. Li and L.F. Tam, [J.D.G. 1992], the dimension of (suitable subspaces of) $L^2\mathcal{H}^1(M)$ reflects in some sense the topology at infinity of the underlying manifold (number of non-parabolic ends). This applies in particular to minimal submanifolds of finite index (where all ends are non-parabolic). Also, on a generic complete manifold, according to the decomposition theorem by Hodge-de Rham-Kodaira, $L^2$ harmonic forms completely represent the (reduced) $L^2$ cohomology of the manifold. It has been recently observed by D. Alexandru-Rugina, [Tensors, 1996], that, in case of manifolds with bounded geometry (e.g. co-compact coverings), the spaces $L^{p<2}\mathcal{H}^k(M)$ imbeds continuously into the corresponding $L^p$ cohomology spaces. Furthermore, it is known from works by J. Dodziuk, [Topology 1977], and V.M. Gol'dshtein, V.I. Kuz'minov, I.A. Shvedov, [Sibirsk. Mat. Zh. 29 1988], that the $L^p$ cohomology (non reduced, in fact) of a co-compact covering is a homotopy invariant of the base (compact!) manifold.

**Theorem 5 (P.-Rigoli-Setti, Rev. Mat. Iberoam.)** *Let* $(M, \langle , \rangle)$ *be a connected, complete,* $m$*-dimensional Riemannian manifold and* $E$ *a Riemannian (Hermitian) vector bundle of rank* $l$ *over* $M$*. The space of its smooth sections is denoted by* $\Gamma(E)$*. Having fixed*

$$a(x) \in C^0(M), \qquad A \in \mathbb{R}, \qquad H \geq p$$

*satisfying the further restrictions*

$$p \geq -A + 1, \quad p > 0, \tag{1}$$

*let* $V = V(a, A, p, H) \subset \Gamma(E)$ *be any vector space with the following property:*

**(P)** *Every* $\xi \in V$ *has the unique continuation property, i.e.,* $\xi$ *is the null section whenever it vanishes on some domain; furthermore the locally-Lipschitz* *function* $u = |\xi|$ *satisfies*

(2)
$$\begin{cases} u\left(\Delta u + a(x)\,u\right) \geq A\,|\nabla u|^2 & \text{weakly on } M \\ \int_{B_r} u^{2p} = o\left(r^2\right) & \text{as } r \rightarrow +\infty. \end{cases}$$

*If there exists a solution* $0 < \varphi \in Lip_{loc}$ *of the differential inequality*

(3)
$$\Delta\varphi + Ha\,(x)\,\varphi \leq 0 \text{ weakly on } M - K$$

*for some compact set* $K \subset M$*, then*

(4)
$$\dim V \leq d,$$

*for some* $d < +\infty$ *depending only on the geometry of* $M$ *in a neighborhood of* $K$*.*

The following consequence extends on previous work by Carron and Li-Wang.

**Corollary 6** *Let* $(M, \langle \, , \, \rangle)$ *be a complete manifold satisfying the global Sobolev inequality*

$$\left(\int |u|^{2\mu}\right)^{\frac{1}{\mu}} \leq S_{\mu}^{-1} \int |\nabla u|^{2}, \ \forall u \in C_{c}^{\infty}(M),$$

*with* $\mu > 1$*. Assume that* $\rho_x \geq -R(x)$ *for some* $R(x) \geq 0$*. Then*

$$R(x) \in L^{\frac{\mu}{\mu-1}}(M) \Longrightarrow \dim L^{2p}\mathcal{H}^{k}(M) < +\infty.$$

*for every* $k = 1, \ldots, m$ *and for every* $p \geq 1$*. Furthermore,*

$$\|R(x)\|_{L^{\frac{\mu}{\mu-1}}(M)} \leq \frac{S_{\mu}}{Cp} \Longrightarrow L^{2p}\mathcal{H}^{k}(M) = 0$$

*with* $C = C(k, m) > 0$ *the constant in the Bochner-Witzenbock formula for harmonic* $k$*-forms.*

# Some Geometric applications

**Theorem 7 (Topology at infinity of submanifolds of CH spaces)** *Let* $f : (M, \langle\ ,\ \rangle) \to (N, (\ ,\ ))$ *be an isometric immersion of a complete manifold* $M$ *of dimension* $m \geq 3$ *into a Cartan-Hadamard manifold* $N$ *whose sectional curvature (along* $f$*) satisfies*

$$(0 \geq)\ {}^{N}Sec_{f(x)} \geq -\ {}^{N}R\,(x)$$

*for some* ${}^{N}R \in C^{0}\,(M)$*. Assume that the mean curvature vector field satisfies* $|H| \in L^{m}(M)$*. Let*

$$a\,(x) = (m-1)\ {}^{N}R\,(x) + |II|\,(|II| + m\,|H|)\,(x)\,.$$

*with* $II$ *the second fundamental tensor of* $f$*. Set* $\mathcal{L} = -\Delta - a(x)$*.*

$$Ind(\mathcal{L}_M) = 0 \Longrightarrow M \text{ has only one end.}$$

$$Ind(\mathcal{L}_M) < +\infty \Longrightarrow M \text{ has finitely many ends.}$$

**Theorem 8 (Reduction of codimension)** *Let $(M, \langle\,,\rangle)$ be a complete Riemannian manifold of dimension $m \geq 3$. Assume*

$$Ricci \geq -R(x) \quad \text{on } M.$$

*Set ${}^{H}\mathcal{L} = -\Delta - HR(x)$. If*

$$Ind\left({}^{H}\mathcal{L}_M\right) < +\infty$$

*for some $H \geq \frac{m-2}{m-1}$, then, there exists a compact set $\mathcal{K} \subset M$ and an integer $N = N(H, \mathcal{K}) \geq m$ such that the following holds.*

*Let $f : M \to \mathbb{R}^d$, $d > N$ be a harmonic immersion whose energy density satisfies the growth condition*

$$\int_{B_R} |df|^{2p} = o\left(R^2\right), \text{ as } R \to +\infty,$$

*for some $\frac{m-2}{m-1} \leq p \leq H$. Then, there is an $N$-dimensional affine subspace $\mathbb{A}^N$ of $\mathbb{R}^d$ such that $f(M) \subset \mathbb{A}^N$.*

## Outline of the proof of the vanishing theorem

Combine the solutions $\psi$ and $\varphi$ so to obtain a new function

$$u = \varphi^{-\frac{p}{H}} \psi^{p}$$

which, in turn, satisfies the easy to handle inequality

$$u \mathrm{div} \left( \varphi^{\frac{2p}{H}} \nabla u \right) \geq 0.$$

Note that

$$\varphi^{\frac{2p}{H}} u^{2} \in L^{1}(M).$$

Now, the key step is to obtain a general Liouville theorem for (possibly changing sign!) solutions of the problem

$$\begin{cases} v \mathrm{div}(w \nabla v) \geq 0 \\ w |v|^{q} \in L^{1}(M). \end{cases}$$

In our special case, deduce $u \equiv$const. Equivalently

$$C\varphi = \psi^H.$$

Use this information into inequality (*) and get

$$H\,(H - 1 + A)\,\psi^{H-2}\,|\nabla\psi|^2 \leq 0.$$

This latter produces the dichotomy in the statement of the theorem and gives the desired conclusions.

# Outline of the proof of the finiteness theorem.

Choose $R >> 1$ in such a way that $K \subset B_R(o)$ and, therefore, inequality (3) holds on $M - B_R(o)$. Note that, by unique continuation, the restriction map

$$\begin{array}{ccc} V & \longrightarrow & \Gamma\left(E|_{B_R}\right) \\ \xi & \longmapsto & \xi|_{B_R} \end{array}$$

is an injective homomorphism. Use the same symbol $V$ to denote the image of $V$ in $\Gamma\left(E|_{B_R}\right)$. An extension of a classical result by P. Li states that if $T \subset V$ be any finite dimensional subspace then, there exists a (non-zero) section $\bar{\xi} \in T$ such that, setting $\bar{\psi} = \left|\bar{\xi}\right|$, it holds

$$\text{(5)} \quad (\dim T)^{\min(1,p)} \int_{B_R} \bar{\psi}^{2p} \leq vol\,(B_R) \min\{l, \dim T\}^{\min(1,p)} \sup_{B_R} \bar{\psi}^{2p}.$$

Now, observe that, on every sufficiently small closed ball,

$$\lambda_1 \left( {}^{H}\mathcal{L}_{B_\delta(x)} \right) > 0,$$

where ${}^{H}\mathcal{L} = -\Delta - Ha(x)$, and therefore there exists $w > 0$ solution of

$$\Delta w + Ha(x) w = 0.$$

As above deduce that

$$u = w^{-\frac{p}{H}} \psi^p$$

satisfies

$$u \operatorname{div} \left( w^{\frac{2p}{H}} \nabla u \right) \geq 0 \text{ on } B_\delta(x).$$

Obtain a local $L^q$-mean value inequality for solutions $u$ of this inequality and apply it to get

$$\sup_{B_\delta} \psi^{2p} \leq C \int_{B_{2\delta}} \psi^{2p}.$$

The local inequalities patches together and, in the special case of $\bar{\psi}$, give

$$\sup_{B_R} \bar{\psi}^{2p} \leq C' \int_{B_{R+1}} \bar{\psi}^{2p}.$$

Inserting into (5) we obtain

(6)

$$(\dim T)^{\min(1,p)} \int_{B_R} \bar{\psi}^{2p}$$

$$\leq C vol\,(B_R) \min\{l, \dim T\ \}^{\min(1,p)} \left\{ \int_{B_R} \bar{\psi}^{2p} + \int_{A(R,R+1)} \bar{\psi}^{2p} \right\}$$

where $A\,(a, b)$ is the annulus $B_b - B_a$. Now, using once again the combination of $\psi$ and $\varphi$ and a careful cut-off analysis inspired by Li-Wang estimating technique obtain the estimate

$$\int_{A(R,R+1)} \bar{\psi}^{2p} \leq C'' \int_{B_R} \bar{\psi}^{2p}.$$

Inserting into (6) and simplifying gives

$$\mathsf{dim}\, T \leq C''' \,\mathsf{min}\, \{l, \mathsf{dim}\, T\ \}$$

for some $C'''$ depending only on the geometry of $B_R$. This proves that any finitely generated subspace $T$ of $V$ have a dimension which is bounded by a universal constant, depending only on the rank $l$ of $E$ and on the geometry of $B_R$. The same bound must work for the dimension of the whole $V$.