Some topics in the theory of harmonic functions on Riemannian manifolds

Stefano Pigola

Bicocca, July 12, 2006

Abstract

In this talk we take a look at some classical and some new results on the interplay between the vanishing and the finite-dimensionality of spaces of special harmonic functions on one hand, and the geometry of the underlying complete Riemannian manifold on the other hand.

Contents

1 Prelude

In what follows $(M, \langle , \rangle)$ always denote a connected, geodesically complete Riemannian manifold (without boundary) of dimension $m \geq 2$. Its Levi-Civita connection is denoted by D. Moreover, we set

$$Riem\,(X,Y)\,Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z$$

for its Riemann curvature tensor and

$$Sec\,(X \wedge Y) = \frac{\langle Riem\,(X,Y)\,Y, X\rangle}{|X|^2\,|Y|^2 - \langle X, Y\rangle^2}$$

for the sectional curvature along the 2-plane Π spanned by the linearly independent vector fields X, Y. The Ricci curvature of M is denoted by

$$\begin{aligned} Ric\,(Y,Z) &= trace_{\langle,\rangle}\,(X \to Riem\,(X,Y)\,Z) \\ &= \sum_{j=1}^{m} \langle Riem\,(E_j, Y)\,Z, E_j \rangle \end{aligned}$$

where $\{E_i\}$ is any o.n. frame.

The Riemannian metric $\langle,\rangle$ of M gives rise to a natural intrinsic distance $dist$ and a natural Riemannian measure $dvol$ which give M the structure of a metric measure space. The distance is defined by

$$dist\,(x,y) = \inf_{\gamma} length\,(\gamma)\,,$$

the infinmum being taken with respect to any piece-wise smooth curve γ from x to y. Since $(M, \langle,\rangle)$ is geodesically complete, according to the Hopf-Rinow Theorem, $(M, dist)$ is a complete metric space with the Heine-Borel property. Thus, setting

$$B_R\,(o) = \{x \in M : dist\,(o,x) < R\}$$

and

$$\partial B_R\,(o) = \{x \in M : dist\,(o,x) = R\}$$

for the metric ball and sphere centered at $o \in M$ and of radius $R > 0$, we have that both $\overline{B_R\,(o)}$ and $\partial B_R\,(0)$ are compact subsets of M. The volume of balls $vol\,(B_R)$ and the area of spheres $vol\,(\partial B_R)$ can be computed by means of the Riemannian measure that, in local coordinates $x^1, ..., x^m$, writes as

$$dvol = \sqrt{G} dx^1 ... dx^m$$

where

$$G = \det\left(\langle \partial_i, \partial_j \rangle\right).$$

Now, let $u : M \to \mathbb{R}$ be a given function. Recall that the **gradient** of u is the vector field ∇u on M defined by

$$\langle \nabla u, X \rangle = X\,(u)$$

for any vector field X on M. Said differently, ∇u is the $\langle,\rangle$-dual of the differential du. The **Hessian** of u is the symmetric 2-tensor defined by

$$Hess\,(u)\,(X,Y) = \langle D_X \nabla u, Y \rangle\,.$$

Finally, the **Laplacian** of u is the divergence-form, linear operator given by

$$\begin{aligned} \Delta u &= trace_{\langle,\rangle} Hess\,(u) \\ &= \sum_{j=1}^{m} Hess\,(u) \left(D_{E_j} \nabla u, E_j\right) \\ &= div\,(\nabla u)\,. \end{aligned}$$

Direct computations show that, in local coordinates $x^1, ..., x^m$,

$$\begin{aligned}\nabla u &= g^{ki} \partial_k u \ \partial_i \\ Hess\,(u)\,(\partial_i, \partial_j) &= \frac{1}{\sqrt{G}} g_{jk} \partial_i \left(\sqrt{G} g^{kl} \partial_l u\right) \\ \Delta u &= \frac{1}{\sqrt{G}} \partial_j \left(\sqrt{G} g^{jl} \partial_l u\right),\end{aligned}$$

where

$$\begin{aligned}g_{ij} &= \langle \partial_i, \partial_j \rangle \\ G &= \det\,(g_{ij}) \\ \left(g^{ij}\right) &= (g_{ij})^{-1}.\end{aligned}$$

In particular

$$\Delta u = g^{jl} \partial_j \partial_l u + \text{first-order terms}$$

is a second order, elliptic operator. For those who prefer the moving frame notation, having fixed a local o.n. coframe $\left\{\theta^j\right\}$, we have,

$$\begin{aligned}du &= u_i \theta^i \\ Hess\,(u) &= u_{ij} \theta^i \otimes \theta^j,\ u_{ij} \theta^j = du_i - u_k \theta^k_i \\ \Delta u &= \sum_j u_{jj}.\end{aligned}$$

Here, $\left\{\theta^i_j\right\}$ are the connection forms defined by the first Cartan equations

$$\begin{cases} d\theta^i = -\theta^i_j \wedge \theta^j \\ \theta^i_j + \theta^j_i = 0. \end{cases}$$

Thus, the function $u : M \to \mathbb{R}$ is said to be **harmonic** if it satisfies

$$\Delta u = 0.$$

When the equality is replaced by the inequality

$$\Delta u \leq 0$$

we say that u is **superharmonic.** Finally reversing the inequality

$$\Delta u \geq 0$$

we get a **subharmonic** function.

Easy application of the strong maximum principle (alternatively you can apply the divergence theorem) shows that:

Fact 1 *Let $(M, \langle , \rangle)$ be compact. Then every harmonic (subharmonic, superharmonic) function is constant.*

A question arises naturally. Let $(M, \langle , \rangle)$ be a generic non-compact manifold; does M support any non-constant harmonic function?

The next result answers the question in the affirmative by showing that, in fact, there are "many" harmonic functions on M.

Theorem 2 (Greene-Wu, [9], [10]) *Let $(M, \langle , \rangle)$ be a non-compact Riemannian manifold. Then there exists a proper imbedding $f = \left(f^1, ..., f^{2m+1}\right) : M \to \mathbb{R}^{2m+1}$ whose components $f^A : M \to \mathbb{R}$ are harmonic functions.*

Remark 3 (minimality) Greene-Wu result is a refined version of the celebrated imbedding theorem by H. Whitney. We point out that, in general, the harmonic imbedding f is not isometric. Namely, f induces on M a metric which may be different from the background metric:

$$f^* can_{\mathbb{R}^{2m+1}} \neq \langle , \rangle .$$

A harmonic, isometric immersion $f : M \to \mathbb{R}^N$ is called a minimal immersion. In general a complete manifold M cannot be minimally immersed into any Euclidean space.

Remark 4 (exhaustion) As a consequence of the Theorem we have that there is no obstruction for a generic manifold to support a (strictly) subharmonic exhaustion function. Simply take $\tau = \sum \left(f^A\right)^2$.

Remark 5 (non-linear curiosity) A non-linear generalization of the Laplace operator is represented by the $p\,(> 1)$-laplacian

$$\Delta_p u = div\left(|\nabla u|^{p-2} \nabla u\right).$$

It is not known whether a generic complete manifold $(M, \langle , \rangle)$ supports any p-harmonic function. The most general result (avoiding curvature conditions) requires that M is disconnected at infinity (see below for the definition).

Thus, every non-compact $(M, \langle , \rangle)$ supports a lot of harmonic functions, independently of its geometry. The geometry of M (volume growth, curvature bounds, topology etc...) enters the game when we investigate harmonic functions with special properties such as L^p-integrability, given sign (semi-boundedness), finite L^p-energy.

By way of example, Li-Tam, [15], discovered that positive and finite L^2-energy harmonic functions are intimately related to the topology at infinity of the underlying complete manifold $(M, \langle , \rangle)$, i.e., to the number of its ends. Let Ω be a compact domain of M. An **end** of M with respect to Ω is any unbounded connected component of $M - \Omega$. Note that the number of ends with respect to Ω is necessarily finite. Clearly, such a number increases with Ω. If it stabilizes to a certain $b \in \mathbb{N}$ as $\Omega \nearrow M$ we say that M has b ends. Otherwise, M has infinitely many ends. A manifold with at least two ends is said to be **disconnected at infinity.** The following result in the setting of minimal submanifolds gives the flavor of Li-Tam harmonic function theory. We shall come back to it later on.

Theorem 6 (Cao-Shen-Zhu, [5], Li-Wang, [16]) *Let $f : (M^m, \langle , \rangle) \to \mathbb{R}^{m+1}$ be a complete, minimal hypersurface. Set $\mathcal{H}_D^{\infty}(M)$ for the vector space of L^{∞}-harmonic functions u on M with finite Dirichlet integral $|\nabla u| \in L^2$. If $\dim \mathcal{H}_D^{\infty}(M) < +\infty$ then M has a finite number of ends. Moreover*

$$\#ends \leq \dim \mathcal{H}_D^{\infty}(M).$$

The present talk aims to present a brief survey of some classical and some very recent results about the interplay between existence, non-existence and finite dimensionality of special harmonic functions and the geometry of the underlying manifold. Obviously, we will not give a comprehensive view. For instance, we shall not touch important topics such as the L^2-cohomological theory related to harmonic functions with finite L^2-energy, splitting-type theorems, Colding-Minicozzi works on Yau conjecture about harmonic functions of polynomial growth and the subsequent developments, harmonic functions on manifolds with special structures, e.g., Kahler manifolds etc...

2 $L^{p<+\infty}$-harmonic functions

Let us first consider the well known situation of $(\mathbb{R}^m, can)$.

Let u be a harmonic function satisfying $u \in L^p(\mathbb{R}^m)$ for some $1 \leq p < +\infty$. Then, $|u(x)| \geq 0$ is subharmonic and by the standard mean value inequality we have, for any $x \in \mathbb{R}^m$,

$$|u(x)| \leq \frac{\int_{B_R(x)} |u|}{vol(B_R(x))}.$$

Using Hölder inequality, the latter implies the L^p-mean value inequality

$$|u(x)|^p \leq \frac{\int_{B_R(x)} |u(x)|^p}{vol(B_R(x))}.$$

Therefore, taking the limit as $R \to +\infty$, we conclude $u \equiv 0$.

Let us now consider the case of a generic complete, non-compact manifold $(M, \langle , \rangle)$. **In general, without any further requirement on the geometry of M, there is no global(!) L^p-mean value inequality.**

2.1 Yau vanishing result and Li-Schoen counter-examples

In the range $1 < p < +\infty$ we have the following surprising L^p-Liouville type result. The proof is completely elementary and uses just the divergence theorem (in this sense it can be considered a non-linear proof). The reader should compare Yau's argument with those supplied in a paper by Greene-Wu, [11] (where, in fact, they required sectional curvature conditions).

Theorem 7 (Yau, [23]) *Let $(M, \langle , \rangle)$ be any complete Riemannian manifold (no assumptions!). Let u be a harmonic function (positive subharmonic function) satisfying $u \in L^p$, some $1 < p < +\infty$. Then, u is constant. In particular, if $vol(M) = +\infty$, $u \equiv 0$.*

In case $p = 1$, more generally in the range $0 < p \leq 1$, the above result fails to hold. Counter-examples first appeared in a paper by Li-Schoen, [14], and involved negatively curved, complete manifolds with fast sectional curvature decay. Precisely, they constructed 2-dimensional surfaces with

$$Sec \asymp -r(x)^{2+\varepsilon}, \text{ as } r(x) \to +\infty$$

and supporting non-constant L^1-harmonic functions. They also constructed 2-dimensional surfaces with

$$Sec \sim -Cr(x)^{-2}, \text{ as } r(x) \to +\infty$$

for large enough $C > 0$, possessing non-constant L^p-harmonic functions with $0 < p < 1$. Here $r(x) = dist(x, o)$ for some fixed origin $o \in M$.

2.2 Li-Schoen L^p-mean value inequalities

To deal with $0 < p \leq 1$ situations it is quite natural to investigate on the geometric assumptions implying L^p-mean value inequalities.

Theorem 8 (Li-Schoen, [14]) *Let $(M, \langle , \rangle)$ be a complete manifold and $o \in M$ a fixed origin. Assume that*

$$Ric \geq -(m-1)\,k \text{ on } B_R(o),$$

for some $k \geq 0$. Let $v \geq 0$ be a subharmonic function on $B_R(x)$. For any $p > 0$ there exists a constant $C = C(m,p) > 0$ such that

$$\sup_{B_{(1-\tau)R}(o)} v^p \leq \tau^{-C\left(1+\sqrt{k}R\right)} \frac{\int_{B_R(x)} v^p}{vol\,(B_R(o))}$$

for any $\tau \in (0, 1/2)$.

Note that, in case $Ric \geq 0$, a result Calabi and Yau gives $vol\,(B_R(o)) \geq cR$ so that we can take limits as $R \to +\infty$ and immediately conclude $v = 0$. In particular, **a complete manifold of non-negative Ricci curvature supports no, non-constant $L^{0<p<+\infty}$ harmonic functions.** In fact much more holds true and a large amount of negative Ricci curvature is permitted. The proof uses again Li-Schoen mean value inequalities in an essential way but the arguments are now more involved. For instance, and this happens in the proof of case 2. below, one hard problem is to show that volumes are infinite and increase fast enough to compensate the explosion of $\tau^{-C\left(1+\sqrt{k(R)}R\right)}$.

Theorem 9 (Li-Schoen, [14]) *Let $(M, \langle , \rangle)$ be a complete manifold. Let $r(x) = dist\,(x, o)$ for some fixed origin $o \in M$.*

1. *Assume that*
$$Ric \geq -C \frac{(1 + r(x))^2}{\log^\alpha\left(1 + r(x)^2\right)},$$
for $r(x) >> 1$ and for some constants $\alpha, C > 0$. Then, every L^1-harmonic function on M must be constant.

2. *Assume that*
$$Ric \geq -\frac{C}{r(x)^2}$$
for $r(x) >> 1$, and for some $C = C(m) > 0$ sufficiently small. Then, every L^p-harmonic function on M, $0 < p < 1$, must be constant.

3 (Semi-)Bounded harmonic functions

Say that a function u on M is **semibounded** if either $\inf_M u > -\infty$ or $\sup_M u < +\infty$. Again, let us first consider the case of $\mathbb{R}^m$. Suppose that u is a semibounded, harmonic function. For instance, assume $\inf_M u > -\infty$. Take

$$v = u - \inf_{\mathbb{R}^m} u.$$

Since v is harmonic and satisfies

$$\inf_{\mathbb{R}^m} v = 0$$

the well known Harnack inequality tells us that

$$0 \le \sup_{\mathbb{R}^m} v \le c \inf_{\mathbb{R}^m} v = 0.$$

It follows that $u \equiv \inf_{\mathbb{R}^m} u$, that is, every semibounded harmonic function on $\mathbb{R}^m$ is constant.

On a generic complete Riemannian manifold, positive harmonic functions do not enjoy any (global!) Harnack inequality. In a sense, the validity of global Harnack inequalities is a peculiarity of almost-Euclidean spaces. This is true from both the metric (Poincarè+Doubling) and the curvature point of view.

3.1 Parabolicity and volumes

The classical maximum principle asserts that a bounded-above subharmonic function on M which achieves its maximum at some interior point must be constant. It is readily seen that, equivalently, if a non constant u attains its maximum u^* at some $x_0 \in M$ then, for every small $\varepsilon > 0$,

$$\inf_{\{u > u^* - \varepsilon\}} \Delta u < 0.$$

Indeed, let Ω_ε be the connected component of the open set $\{u > u^* - \varepsilon\}$ containing x_0. Thus, Ω_ε is an open set. If $\Delta u \ge 0$ on Ω_ε, then the maximum principle forces $u = u^*$ on Ω_ε. In particular, Ω_ε is closed. By connectedness $\Omega_\varepsilon = M$ and u is a constant function.

Now, suppose u is just bounded above and "achieves its supremum at infinity". What can we say about Δu? Answering this question leads to the concept of parabolic (and stochastically complete) manifolds.

A manifold $(M, \langle , \rangle)$ is said to be **parabolic** if every subharmonic function on M bounded from above must be constant:

$$\begin{cases} \Delta u \ge 0 \\ \sup_M u < +\infty \end{cases} \Longrightarrow u = const.$$

In particular **a parabolic manifold does not support any semibounded harmonic function.**

The concept of parabolic manifold is related to a wide class of equivalent properties of M involving the Green kernel, the linear capacity, the Brownian motion, etc... Here, we limit ourselves to point out the following maximum principle characterization.

Theorem 10 $(M, \langle , \rangle)$ *is parabolic if and only if for every non constant function u satisfying $u^* = \sup_M u < +\infty$ it holds*

$$\inf_{\{u > u^* - \varepsilon\}} \Delta u < 0$$

for every sufficiently small $\varepsilon > 0$.

From the geometric point of view, parabolicity of a complete manifold is tightly related to the growth rate of the volume of geodesic balls. The following result is independently due to Ahlfors and Nevanlinna for surfaces, and to Lyons-Sullivan and Grigor'yan for general Riemannian manifolds; see [8].

Theorem 11 *Let* $(M, \langle , \rangle)$ *be complete. If*

$$t \longmapsto \frac{1}{vol\left(\partial B_t\right)} \notin L^1\left(+\infty\right)$$

then M *is parabolic.*

Thus, for instance, the plane $\mathbb{R}^2$ is parabolic. It can be shown that, for rotationally symmetric Riemannian metrics, the above volume condition is also necessary for the manifold to be parabolic. Thus, $\mathbb{R}^{m\geq 3}$ is not parabolic.

Nonlinear proof of Theorem 11 (Rigoli-Setti, [20]). Let u be a subharmonic function satisfying $u^* = \sup_M u < +\infty$. Up to an additive constant we can assume $u^* = 0$. Consider the vector field $Z = e^u \nabla u$. Apply the divergence theorem on a geodesic ball B_r and elaborate:

$$\int_{\partial B_r} e^u \left|\nabla u\right| \geq \int_{\partial B_r} e^u \left\langle \nabla u, \frac{\partial}{\partial r} \right\rangle = \int_{B_r} divZ \geq \int_{B_r} e^u \left|\nabla u\right|^2.$$

Note that

$$\int_{\partial B_r} e^u \left|\nabla u\right| \leq \left\{\int_{\partial B_r} e^u \left|\nabla u\right|^2\right\}^{1/2} \left\{\int_{\partial B_r} e^u\right\}^{1/2}.$$

Therefore

$$\left\{\int_{\partial B_r} e^u \left|\nabla u\right|^2\right\}^{1/2} \left\{\int_{\partial B_r} e^u\right\}^{1/2} \geq \int_{B_r} e^u \left|\nabla u\right|^2.$$

Set

$$H\left(r\right) = \int_{B_r} e^u \left|\nabla u\right|^2$$

and note that, by the co-area formula,

$$H'\left(r\right) = \int_{\partial B_r} e^u \left|\nabla u\right|^2.$$

The above rewrites

$$\frac{H'\left(r\right)}{H\left(r\right)^2} \geq \frac{1}{\int_{\partial B_r} e^u}.$$

Integrating on $[R, r]$, and recalling that $u \leq 0$, gives

$$\frac{1}{H\left(R\right)} \geq \frac{1}{H\left(R\right)} - \frac{1}{H\left(r\right)} \geq \int_R^r \frac{1}{\int_{\partial B_t} e^u} \geq \int_R^r \frac{1}{vol\left(\partial B_t\right)}.$$

Letting $r \to +\infty$ we conclude $H\left(R\right) = 0$, that is, $\left|\nabla u\right| = 0$ and u is constant. ■

3.2 Structure at infinity

Suppose we are given a complete manifold $(M, \langle , \rangle)$. Denote by $\mathcal{H}^+\left(M\right)$ the vector space spanned by the positive harmonic functions of M. Also, introduce $\mathcal{H}^0\left(M\right)$ the vector space spanned by the set of harmonic functions u with the following property: there is a compact domain Ω such that u is semibounded on each of the (finitely many) ends of M with respect of Ω. Then, we have the following

Theorem 12 (Li-Tam, [15]) *Assume* $\dim \mathcal{H}^0(M) < +\infty$*. Then*

$$\#ends(M) \leq \dim \mathcal{H}^0(M).$$

Moreover, if M *is non-paraobolic and* $\dim \mathcal{H}^+(M) < +\infty$ *then*

$$\#ends(M) \leq \dim \mathcal{H}^+(M).$$

Using this result, Li-Tam obtained finiteness of the number of ends for manifolds with asymptotically-nonnegative Ricci curvature.

Theorem 13 (Li-Tam, [15]) *Let the complete manifold* $(M, \langle , \rangle)$ *satisfy*

$$Ric \geq -k(r(x))$$

for some $0 \leq k \in C^0(\mathbb{R}_+)$ *such that*

$$t^{m-1}k(t) \in L^1(+\infty).$$

Then

$$\#ends(M) < +\infty.$$

3.3 Negatively curved manifolds

It is readily seen that there are negatively curved manifolds supporting a large amount of non-constant, bounded harmonic functions, despite the fact that they possess only one end! (Li-Tam theory is not efficient in negative curvature). For instance, consider the 2-dimensional hyperbolic space of constant Gaussian curvature -1 given by

$$(\mathbb{H}^2, hyp) = (B_1, \rho^2 can_{\mathbb{R}^2}),$$

with *can* the canonical metric of $\mathbb{R}^2 \supset B_1$ and

$$\rho(x) = \frac{2}{1 - |x|^2}.$$

Note that the hyperbolic Laplacian is related to the Euclidean one by

$$\Delta_{\mathbb{H}^2} = \frac{1}{\rho^2(x)} \Delta_{\mathbb{R}^2}.$$

Now, for any $\varphi \in C^0(\partial B_1)$, solve the Dirichlet problem

$$\begin{cases} \Delta_{\mathbb{R}^2} u = 0 & \text{on } B_1 \\ u = \varphi & \text{on } \partial B_1. \end{cases}$$

Then, u is a bounded harmonic function on $\mathbb{H}^2$.

The situation generalizes to (quasi isometric images of) simply connected, complete Riemannian manifolds $(M, \langle , \rangle)$ satisfying

$$-A^2 \leq Sec \leq -B^2 < 0.$$

Anderson, [1], Sullivan, [21] and, with a more direct and simplified argument (which works even in non-linear settings), Anderson-Schoen, [2], proved that we can compactify M via a geometric boundary $S(\infty)$, **the sphere at infinity**, and solve the asymptotic Dirichlet problem as in case of the hyperbolic disk.

Note that, in more recent works, the above curvature conditions have been relaxed. In this respect, we limit ourselves to quote [13].

3.4 Non-negatively curved manifolds

In this realm one meets one of the most fruitful techniques developed by Yau (and Cheng). Manifolds of non-negative Ricci curvature enjoy global Harnack inequalities. Yau obtained such a conclusion as an integrated form of his famous **gradient-estimates for positive harmonic functions.** The latter are proved as a clever application of the usual maximum principle for bounded-above functions attaining their maximum.

Theorem 14 (Yau, [22]) *Let $(M, \langle , \rangle)$ be a complete manifold satisfying*

$$Ric \geq -k$$

for some constant $k \geq 0$. Suppose that u is a positive harmonic function on M. Then, for every geodesic ball $B_R(x)$, it holds

$$\frac{|\nabla u|}{u} \leq C \frac{1 + R\sqrt{k}}{R} \text{ on } B_{R/2}(x)$$

where $C > 0$ is a constant depending only on m. In particular,

$$\sup_{B_{R/2}(x)} u \leq e^{2C\left(1+R\sqrt{k}\right)} \inf_{B_{R/2}(x)} u.$$

Accordingly, **non-negatively Ricci curved manifold do not support any non-constant, semibounded, harmonic function.**

4 Harmonic functions with finite L^p-energy

Once again, let us first consider the Euclidean case $\mathbb{R}^m$. Take a harmonic function u on $\mathbb{R}^m$. Easy computations show that

$$\Delta |\nabla u|^2 = 2 |Hess(u)|^2 .$$

Therefore, if $|\nabla u| \in L^p(\mathbb{R}^m)$ for some $p > 0$, by the mean value inequality applied to $|\nabla u|^2$ we have $|\nabla u| = 0$ and u is a constant function.

What happens on a generic manifold? The following result goes back to Wietzenbock in the 20's and was independently rediscovered and systematically used by Bochner in 50's.

Theorem 15 (Bochner-Weitzenbock formula) *Lat u be a harmonic function on a manifold $(M, \langle , \rangle)$. Then*

$$\frac{1}{2}\Delta |\nabla u|^2 = |Hess(u)|^2 + Ric(\nabla u, \nabla u).$$

*Furthermore the following **refined Kato inequality** holds*

$$|Hess(u)|^2 \geq \left(1 + \frac{1}{m-1}\right) |\nabla |\nabla u||^2 .$$

Thus, in particular, if

$$Ric \geq -q(x)$$

for some function $0 \leq q(x) \in C^0(M)$, one has (in the sense of distributions)

$$|\nabla u| \left(\Delta |\nabla u| + q(x) |\nabla u|\right) \geq \frac{1}{m-1} |\nabla |\nabla u||^2 .$$

It follows from this result and from what we said in the previous sections that, **in case** $Ric \geq$ 0**, there are no non-constant harmonic functions with finite L^p-energy,** $p > 0$**.** This confirms once more the impression that *a* $Ric \geq 0$-manifold really looks like Euclidean space. However, in general, the contribution of the Ricci curvature heavily influences the analysis.

4.1 An example in negative curvature

Suppose we are given a 2-dimensional manifold $(M, \langle,\rangle)$. We set $L^2\mathcal{H}(M)$ for the vector space of L^2 harmonic functions on M. Then $L^2\mathcal{H}(M)$ depends only on the conformal class of $\langle,\rangle$. Indeed, if $\widetilde{\langle,\rangle} = \lambda^2(x)\langle,\rangle$ for some smooth function $\lambda(x) > 0$, we have

$$\int_M |\nabla u|^2 \, dvol = \int_M \left|\widetilde{\nabla} u\right|^2_{\widetilde{\langle,\rangle}} \, d\widetilde{vol},$$

that is, the Dirichlet integral is a conformal invariant. Furthermore, we have already observed that

$$\widetilde{\Delta} = \frac{1}{\lambda^2(x)}\Delta$$

so that harmonicity is a conformal invariant. The claim is proved.

Now, consider the 2-dimensional Hyperbolic space $\left(\mathbb{H}^2, hyp\right) = \left(B_1, \rho^2 can_{\mathbb{R}^2}\right)$ of constant Gaussian curvature -1. Obviously, since B_1 has finite Euclidean volume, we have

$$\dim L^2\mathcal{H}(B_1) = +\infty.$$

It follows that

$$\dim L^2\mathcal{H}\left(\mathbb{H}^2\right) = +\infty.$$

4.2 Spectral assumptions: vanishing and finiteness results

The problem is now to find a way to take into account the contribution of the (negative part of the) Ricci tensor. Assume

$$Ric \geq -q(x)$$

for some function $q(x) \geq 0$. A possible strategy consists in considering spectral properties of the geometric Schrodinger operator

$$\mathcal{L} = -\Delta - q(x)$$

In the geometric realm, this strategy has its roots in the classical theory of minimal (hyper)surfaces of the Euclidean space (where $\mathcal{L} = -\Delta - |II|^2$ is the stability operator).

Recall that the **bottom of the spectrum of $\mathcal{L}$** is defined by

$$\lambda_1^{\mathcal{L}}(M) = \inf_{0 \neq \varphi \in C_c^\infty(M)} \frac{\int_M |\nabla\varphi|^2 - q(x)\varphi^2}{\int_M \varphi^2}.$$

In a 1988 paper, Elworthy-Rosenberg, [6], first observed that on a complete manifold with $Ric \geq -k$, condition

$$\lambda_1^{\mathcal{L}}(M) \geq 0$$

forces the constancy of harmonic functions (vanishing of harmonic forms) with finite L^2-energy. Their technique relies on very refined probabilistic tools. The Ricci curvature assumption is in order to guarantee the completeness of M with respect to the Brownian paths. Soon later, Berard, [3], generalized Elworthy-Rosenberg results by removing the curvature condition. Moreover, his proof is completely elementary and makes a direct use of the spectral assumption. His direct approach, however, forces him to consider only L^2-energies. What about other energies?

It should be noted that one can also be given a manifold $(M, \langle , \rangle)$ such that the above spectral property is satisfied just outside some compact set $K \subset M$, namely,

$$\lambda_1^{\mathcal{L}}(M - K) \geq 0.$$

Think for instance of a condition like $Ric \geq 0$ on $M - K$. Classical works by (Gulliver and) Fischer-Colbrie, [7], in minimal surface theory, show that this condition is related (implied) by the **finiteness of the (generalized) Morse index** $Ind(\mathcal{L}, M)$ of $\mathcal{L}$. Recall that

$$Ind(\mathcal{L}, M) = \sup_{\substack{\Omega \subset\subset M \\ \partial\Omega \in C^\infty}} Ind(\mathcal{L}_\Omega)$$

where $\mathcal{L}_\Omega$ denotes the Friedrichs extension of $(\mathcal{L}, C_c^\infty(\Omega))$ and $Ind(\mathcal{L}_\Omega)$ is the (finite) number, counting multiplicities, of negative eigenvalues of $\mathcal{L}_\Omega$. Carron, [4], (in the presence of a Sobolev type inequality) and Li-Wang, [17], (general case) recently obtained that this weaker spectral condition gives finite dimensionality results for the space of harmonic functions (forms) with finite L^2-energy.

Very recently, using quite different techniques (that allowed us to avoid the use of refined Kato inequalities), we were able to obtain extensions of both vanishing and finiteness result in the L^p-case. In fact, our results apply to more general geometric situations which are governed by a Bochner-type differential inequality. In the following statement, for any $q \geq 1$, $L^q\mathcal{H}(M)$ stands for the vector space of harmonic functions u satisfying $|\nabla u| \in L^q$.

Theorem 16 (P.-Rigoli-Setti, [18], [19]) *Let $(M, \langle , \rangle)$ be a complete manifold of dimension $m > 3$ satisfying*

$$Ric \geq -q(x)$$

for some $0 \leq q(x) \in C^0(M)$. Assume that, for some $H \geq \frac{m-2}{m-1}$

$$\lambda^{-\Delta - Hq(x)}(M) \geq 0.$$

Then, for every $\frac{m-2}{m-1} < p \leq H$

$$\dim L^{2p}\mathcal{H}(M) = 1.$$

In case

$$\lambda^{-\Delta - Hq(x)}(M - K) \geq 0$$

for some compact set $K \subset M$, then there is a constant $N \geq m$ depending only on H and on the geometry of M in a neighborhood of K such that

$$\dim L^{2p}\mathcal{H}(M) \leq N < +\infty,$$

for every $\frac{m-2}{m-1} \leq p \leq H$.

Remark 17 In case $m \leq 3$, the result still holds assuming that $1/2 < p \leq H$.

By way of example, let us consider the following consequence. By Greene-Wu theorem, every manifold $(M, \langle , \rangle)$ imbeds into some Euclidean space via a harmonic immersion f. Greene-Wu did not introduce any control on the energy density

$$|df|^2 = tr_{\langle , \rangle} f^* can$$

of the immersion. Such a request could require higher dimensional Euclidean ambient spaces. . We have the following Bernstein-type result.

Theorem 18 *Let $(M, \langle , \rangle)$ be a complete manifold satisfying, for some $H \geq \frac{m-2}{m-1} > \frac{1}{2}$ and some compact set $K \subset M$,*

$$\lambda^{-\Delta - Hq(x)} (M - K) \geq 0,$$

where $q(x)$ denotes the pointwise lower Ricci curvature eigenvalue. Then, there is a constant $N \geq m$ depending only on H and on the geometry of M in a neighborhood of K such that the following holds. Let $f : M \to \mathbb{R}^{T>N}$ be a harmonic immersion whose energy density

$$|df|^2 = tr_{\langle , \rangle} f^* can \in L^p(M)$$

for some $\frac{m-2}{m-1} \leq p \leq H$. Then $f(M) \subset \mathbb{A}^N$, an N-dimensional affine subspace of $\mathbb{R}^T$.

4.3 Structure at infinity

Harmonic functions with finite Dirichlet integral can be used to count special ends of the manifold. Suppose we are given an end E, with smooth boundary ∂E, of the complete manifold $(M, \langle , \rangle)$. Say that **the end E is parabolic** if the double of E is a parabolic manifold (without boundary). Similarly, one defines non-parabolic ends. A nice result by Cao-Shen-Zhu, [5], gives the following description of non-parabolic ends in the presence of Sobolev-type inequalities.

Theorem 19 (Cao-Shen-Zhu, [5]) *Suppose $(M, \langle , \rangle)$ supports the L^2-Sobolev inequality*

$$\left\{ \int_M u^{2\mu} \right\}^{\frac{1}{\mu}} \leq S_2 \int_M |\nabla u|^2, \ \forall u \in C_c^\infty(M),$$

for some $\mu > 1$ and some $S_2 > 0$. Let E be an end of M. Then either E has finite volume or E is non-parabolic.

In particular, assume that $(M, \langle , \rangle)$ supports an L^1-Sobolev inequality (hence an L^2 one)

$$\left\{ \int_M |u|^{\mu} \right\}^{\frac{1}{\mu}} \leq S_1 \int_M |\nabla u|, \ \forall u \in C_c^\infty(M),$$

for some $\mu > 1$ and $S_1 > 0$. A classical result by Federer-Fleming states that the latter is equivalent to the isoperimetric inequality

$$\{vol\Omega\}^{\frac{1}{\mu}} \leq S_1 vol\partial\Omega$$

which in turn, once integrated, implies

$$volB_r(x) \geq Cr^{\frac{1}{\mu}+1},$$

with $C > 0$ an absolute(!) constant. It follows that every end of M has infinite volume and, therefore, is non-parabolic. By way of example, we know from Hoffman-Spruck, [12], that a minimal submanifold of a Cartan-Hadamard space enjoys an L^1-Sobolev inequality. In fact, according to Carron, see [4] and references therein, the same is true if we replace minimality with the assumption that the mean curvature H of the submanifold satisfies $|H| \in L^m(M)$. Therefore, in this Setting, only non-parabolic ends appear.

The theory of Li-Tam alluded to in the previous sections also relates non-parabolic ends to harmonic functions with finite Dirichlet integral.

Theorem 20 (Li-Tam, [15]) *Let $(M, \langle , \rangle)$ be a complete manifold. Set $\mathcal{H}^{\infty}_{D}(M)$ for the vector space of L^{∞}-harmonic functions u on M with finite Dirichlet integral $|\nabla u| \in L^2$. If $\dim \mathcal{H}^{\infty}_{D}(M) < +\infty$ then M has a finite number of non-paraobolic ends. Moreover*

$$\#\textit{non-parabolic ends}\,(M) \leq \dim \mathcal{H}^{\infty}_{D}(M)\,.$$

Obviously $\mathcal{H}^{\infty}_{D}(M) \subseteq L^2\mathcal{H}(M)$ and, therefore, we can conclude

Proposition 21 *Let $(M, \langle , \rangle)$ be a complete manifold of dimension $m \geq 2$ satisfying*

$$Ric \geq -q(x)$$

for some $0 \leq q(x) \in C^0(M)$. Assume that

$$\lambda^{-\Delta - q(x)}(M) \geq 0.$$

Then, M has only one non-parabolic end. Furthermore, in case

$$\lambda^{-\Delta - q(x)}(M - K) \geq 0$$

for some compact set $K \subset M$, then M has finitely many non-parabolic ends.

References

[1] M.T.Anderson, *The Dirichlet problem at infinity for negatively curved manifolds.* J. Diff. Geom. **18** (1983), 701–721

[2] M.T. Anderson, R. Schoen, *Positive harmonic functions on complete manifolds of negative curvature.* Ann. of Math. **121** (1985), 429–461.

[3] P. Bérard, *A note on Bochner type theorems for complete manifolds.* Manuscripta Math. **69** (1990), no. 3, 261–266.

[4] G. Carron, *L^2-Cohomolgie et inégalités de Sobolev.* Math. Ann. **314** (1999), 613–639.

[5] H.-D. Cao, Y. Shen, S. Zhu, *The structure of stable minimal hypersurfaces in $\mathbb{R}^{m+1}$.* Math. Res. Lett. **4** (1997), 637–644.

[6] K.D. Elworthy, S. Rosenberg, *Generalized Bochner theorems and the spectrum of complete manifolds.* Acta Appli. Math. **12** (1988), 1–33.

[7] D. Fischer-Colbrie, *Complete minimal surfaces with finite Morse index in three manifolds.* Invent. Math. **82** (1985), 121–132.

[8] A. Grigor'yan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds.* Bull. Amer. Math. Soc.

[9] R. Greene, H.H. Wu, *Embeddings of open Riemannian manifolds by harmonic functions.* Ann. Inst. Fourier, Grenoble **25** (1975), 215–235.

[10] R. Greene, H.H. Wu, *Whitney's imbedding theorem by solutions of elliptic equations and geometric consequences.* Proc. Symp. Pure Math. **27** (1975), 287–296.

[11] R. Greene, H.H. Wu, *Integrals of subharmonic funtions on manifolds of nonnegative curvature.* Invent. Math. **27**(1974), 265–298.

[12] D. Hoffman, J. Spruck, *Sobolev and isoperimetric inequalities for Riemannian submanifolds.* Comm. Pure Appl. Math. **27** (1974), 715–727.

[13] E. Leclercq, *The asymptotic Dirichlet problem with respect to an elliptic operator on a Cartan-Hadamard manifold with unbounded curvatures.* C.R. Acad. Sci. Paris **325** (1997), 857–862.

[14] P. Li, R. Schoen, *L^p and mean value properties of subharmonic functions on Riemannian manifolds.* Acta Math. **153** (1984), no. 3-4, 279–301.

[15] P. Li, L.F. Tam, *Harmonic functions and the structure of complete manifolds.* J. Differential Geom. **35** (1992), no. 2, 359–383.

[16] P. Li, J. Wang, *Minimal hypersurfaces of finite index.* Math. Res. Let. **9** (2002), 95–103.

[17] P. Li, J. Wang, *Stable minimal hypersurfaces in a nonnegatively curved manifold.* J. reine angew. Math. **566** (2004), 215–230.

[18] S. Pigola, M. Rigoli, A.G. Setti, *Vanishing theorems on Riemannian manifolds and applications.* J. Funct. Anal. **229** (2005), 424–461.

[19] S. Pigola, M. Rigoli, A.G. Setti, *A finiteness result for the space of L^p harmonic sections.* Preprint.

[20] M. Rigoli and A.G. Setti, *Liouville-type theorems for φ-subharmonic functions*, Rev. Mat. Iberoamericana, **17** (2001), 471–520.

[21] D. Sullivan, *The Dirichlet problem at infinity for negatively curved manifolds.* J. Diff. Geom. **18** (1983), 723–732.

[22] S.T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228.

[23] S.T. Yau, *Some function-theoretic properties of complete Riemannian manifold and their applications to geometry.* Indiana Univ. Math. J. **25** (1976), 659–670.