Geometric aspects of the p-Laplacian on complete manifolds

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I. Introductory examples

We are given an m-dimensional Riemannian manifold $(X^m, \langle , \rangle)$. A natural way to detect the geometry and the topology of X is to view X either as the domain or as the target space of some interesting class of maps. Clearly, the Riemannian structure adds information on X and therefore the interesting maps should take them into account.

Let us consider a couple of (classical) examples to give some flavour of ideas and techniques and to introduce (some of) the main ingredients.

Let M, N be compact, with $Sec_N \leq 0$. Let $f : M \to N$ be a smooth map. Then we have

Th. 1 (Eells-Sampson, Hartman)

$$\exists u : M \to N : \int_M |du|^2 = \min \left\{ \int_M |dh|^2 : h \textit{ homotopic to } f \right\}.$$

The minimizer u satisfies the (system of) equations

$$\Delta u := \operatorname{div}(du) = 0$$

i.e. u is a harmonic map. Note: u is smooth by elliptic regularity. In particular, the validity of a Liouville type result

$$\Delta u = 0 \Longrightarrow u = \text{const}$$

gives that f is topologically trivial. For instance, we have the following

Th. 2 (Eells-Sampson) *M cmpt, $Ric_M \geq 0$ and N cmpt, $Sec_N \leq 0$.*

(a) If $Ric_M(p_0) > 0$ for some $p_0 \in M \Longrightarrow$ Liouville for harmonic maps $\Longrightarrow$ every smooth $f : M \to N$ is homotopically trivial.

(b) If $Sec_N < 0$ then either the harmonic map $u : M \to N$ is constant or $u(M) = \Gamma$ a closed geodesic of N.

Proof. Let $u : M \to N$ be harmonic. The Bochner-Weitzenböck formula states

$$\frac{1}{2}\Delta |du|^2 = |Ddu|^2 + \sum_i \langle du(Ric_M(E_i)), du(E_i)\rangle - \sum_{i,j} Sec_N(du(E_i) \wedge du(E_j)) \left|du(E_i) \wedge du(E_j)\right|^2 .$$

Since $Ric_M \geq 0$ and $Sec_N \leq 0$,

$$\Delta |du|^2 \geq 0,$$

equality holding iff $Ddu = 0$. Use Stokes theorem with $X = |du|^2 \nabla |du|^2$:

$$0 = \int_M \mathrm{div}(X) \geq \int_M \left|\nabla |du|^2\right|^2 \geq 0 \Rightarrow |du| \equiv \textbf{const}.$$

and du is parallel. If $Ric_M(p_0) > 0$ then $d_{p_0}u = 0$ and this implies $du = 0$. Similarly if $Sec_N < 0$ and $du \neq 0$, since $du(E_i) \wedge du\left(E_j\right) = 0$ we obtain that $u(M)$ is 1-dimensional. Since $Ddu = 0 \Rightarrow u$ maps geodesics into geodesics$\Rightarrow u(M) \subset \Gamma$ geodesic. Assume Γ simple, otherwise more tricky. If Γ is not closed then u is homotopically trivial. But (M cmpt) it can be shown that u minimizes energy in its homotopy class $\Rightarrow u \equiv$ **const**. Contradiction. It is now easy to obtain $u(M) = \Gamma$. ∎

Now, some classical applications.

Application I. We first illustrate a use of X as a target space.

Th. 3 (Preissman) *X cmpt, $Sec < 0$. Then $\mathbb{Z}^2 \not\subset \pi_1(X)$.*

Proof. By contradiction, $\mathbb{Z}^2 \subset \pi_1(X)$. Fix any injective homomorphism $\rho : \pi_1(T^2) \simeq \mathbb{Z}^2 \to \pi_1(X)$ with T^2 the flat torus. Since $Sec_X \leq 0$, by the general theory of aspherical spaces, we can assume that $\exists$smooth nonconst map $u : T^2 \to X$ which induces ρ up to some $\alpha \in \mathrm{Aut}(\pi_1(X))$, say $\alpha \circ \rho = u_{\#}$. By Eells-Sampson-Hartman, we can take u harmonic. Liouville Theorem$\Rightarrow$ $u(T^2) =$closed geodesic of X. Therefore, $u_{\#}$ maps the generators of $\pi_1(T^2)$ onto a single loop$\Rightarrow$ $u_{\#}$ is not injective. Contradiction. ∎

The flat-torus theorem by Lawson-Yau and Gromoll-Wolf can be obtained along the same line.

Application II. Now we illustrate a use of X as a source space.

Th. 4 *Let X be cmpt with $Ric_X \geq 0$ and $Ric_X > 0$ somewhere. Then, every homomorphism $\rho : \pi_1(X) \to \pi_1(N)$ where N cmpt and $Sec_N \leq 0$, must be trivial: $\rho \equiv 1$.*

Proof. As above, we can assume that $\exists$smooth harmonic map $u : X \to N$ such that $\alpha \circ \rho = u_\#$, for some $\alpha \in \mathrm{Aut}(\pi_1(N))$. Since $Ric_X > 0$ at some $x_0 \in X$, by the Liouville thm u is constant$\Rightarrow u_\# \equiv 1 \Rightarrow \rho \equiv 1$. $\blacksquare$

A consequence. *There is no metric g on $\mathbb{R}^m$ s.t.: (a) $g = g_{Eu}$ on $\mathbb{R}^m \backslash \mathbb{B}_1(0)$; (b) $Ric_g \geq 0$ on $\mathbb{R}^m$; (c) $Ric_g > 0$ at some $x_0 \in \mathbb{B}_1(0)$.*

Cut a cube around $\mathbb{B}_1(0)$, periodise it to get an m-torus X with $Ric_X \geq 0$ and $Ric(x_0) > 0$. Let N be the flat torus. By the thm, the homomorphism $id : \pi_1(X) \to \pi_1(N)$ is trivial. Contradiction. (**Remark.** By Lohkamp, there exist $Ric < 0$ balls!!!)

II. p-harmonic functions and maps

The previous examples involve (2-)harmonic maps. The concept was introduced by Eells-Sampson in the mid '60s and extends the notion of harmonic function.

Let $u : (M^m, \langle , \rangle_M) \to (N^n, \langle , \rangle_N)$ be a smooth map. The Hilbert-Schmidt norm of its differential $du \in \Gamma\left(T^*M \otimes u^{-1}TN\right)$ is denoted by $|du|$. Let $p > 1$.

Def. 1 *The map u is said to be* <u>*p-harmonic*</u> *if*

$$\Delta_p u := \operatorname{div}\left(|du|^{p-2}\, du\right) = 0,$$

where $-\operatorname{div}$ is the formal adjoint of d with respect to the standard L^2-inner product on vector valued 1-forms. The operator $\Delta_p u$ is called the <u>*p-Laplacian*</u> *(or* <u>*p-tension field*</u>*) of u.*

In case $u \in C^1$ the above condition has to be interpreted in the sense of distributions, i.e.,

$$(\Delta_p u, \eta) = -\int_M \left\langle |du|^{p-2}\, du, d\eta \right\rangle = 0,$$

$\forall \eta \in \Gamma_c(u^{-1}TN)$. In local coordinates the above writes

$$-\int |\partial \overrightarrow{u}|^{p-2} \left\{ \langle \partial \overrightarrow{u}, \partial \overrightarrow{\eta} \rangle + \left\langle \overrightarrow{\Gamma}\left(\partial \overrightarrow{u}, \partial \overrightarrow{u}\right), \overrightarrow{\eta} \right\rangle \right\} = 0,$$

where $\overrightarrow{\Gamma}$ is an $\mathbb{R}^n$-valued quadratic form (involving ${}^N\Gamma^A_{BC}$). Note also the relation between Δ_p and Δ:

$$\Delta_p u = |du|^{p-2}\, \Delta u + du \left(\nabla\, |du|^{p-2} \right).$$

In the special case $N = \mathbb{R}$ one can also speak of <u>p-subharmonic function</u> whenever $\Delta_p u \geq 0$ and of <u>p-superharmonic function</u> if $\Delta_p u \leq 0$.

II.a. p-harmonic maps as "canonical" representatives

We are interested in complete non-compact domains. It is then natural to prescribe asymptotic (decay) properties to maps, more precisely on the energy of maps. Say that $f : M \to N$ has finite p-energy if $|df|^p \in L^1(M)$. According to results by R. Schoen and S.T. Yau, F. Burstall, B. White, S.W. Wei, p-harmonic maps can be considered as canonical representatives of homotopy class of maps with finite p-energy into nonpositively curved targets.

Th. 5 *Let $(M, \langle , \rangle_M)$ be complete and $(N, \langle , \rangle_N)$ be compact with $Sec_N \leq 0$. Fix a smooth map $f : M \to N$ with finite p-energy $|df|^p \in L^1(M)$, $p \geq 2$. Then, in the homotopy class of f, there exists a p-harmonic map $u \in C^{1,\alpha}(M, N)$ with $|du|^p \in L^1(M)$. If $p = 2$ then $u \in C^\infty(M, N)$.*

Some consequences and questions that arise naturally from the existence thm:

(a) **Trivial homotopy type**. Liouville type thms under geometric assumptions on $M \Rightarrow$ a map $f : M \to N$ with finite p-energy must be topologically trivial.

(b) **Comparison of homotopic p-harmonic maps**. How many p-harmonic maps with finite p-energy are there in a given homotopy class ?

In case $p = 2$ (harmonic case) both questions in the non-compact setting are answered in deep seminal works by Schoen-Yau (the compact case is due to P. Hartman). They proved:

(A) vanishing results for harmonic maps assuming that either $Ric_M \geq 0$ or M is a stable minimal hypersurface in $\mathbb{R}^{m+1}$;

(B) comparison of homotopic harmonic maps and uniqueness of the harmonic representative, assuming $\mathrm{vol}\,(M) < +\infty$.

II.b. Vanishing for p-harmonic maps

Schoen-Yau vanishing results alluded to in (A) are unified and extended by allowing a controlled amount of negative Ricci curvature (and different energies).

The negative part of the curvature is measured via a spectral assumption. Suppose $Ric_M \geq -a(x)$, $a(x) \geq 0$. Let

$$\mathcal{L}_H = -\Delta - Ha(x),$$

$H \in \mathbb{R}$ is a parameter. By definition

$$\lambda_1(\mathcal{L}_H) := \inf \left\{ \frac{\int |\nabla\varphi|^2 - Ha(x)\varphi^2}{\int \varphi^2} : \varphi \in C_c^\infty(M) \setminus \{0\} \right\}.$$

Intuitively, $\lambda_1(\mathcal{L}_H) \geq 0$ relies on the fact that $a(x)$ is small in some integral sense. In the terminology of P. Li and J. Wang, $\lambda_1(\mathcal{L}_H) \geq 0 \iff$ a weighted Poincaré inequality holds.

Th. 6 *Let M be complete, noncmpt, $Ric \geq -a(x)$ with $\lambda_1(\mathcal{L}_H) \geq 0$ for some $H > (m-1)/m$. Let N be complete, $Sec_N \leq 0$. Then every harmonic map $u : M \to N$ with finite energy $|du| \in L^2$ must be constant. In particular, every map $f : M \to N$ with $|df| \in L^2$ is homotopically trivial.*

Rmk 1 $a(x) \geq 0 \Rightarrow \lambda_1(\mathcal{L}_H) \geq 0$ is weaker than $\lambda_1(\mathcal{L}_1) \geq 0$ previously considered e.g. by [P.-Rigoli-Setti, JFA '05].

Proof. Starting point: Bochner formula+refined Kato (RHS)

$$\boxed{|du|\,\Delta\,|du| + \mathrm{a}(x)\,|du|^2} = |Ddu|^2 - |\nabla\,|du||^2 \boxed{\geq \frac{1}{m}\,|\nabla\,|du||^2.}$$

By applying the next vanishing result with $\psi = |du|$, $A = -1/m$ and $p = 2$ we deduce that $|du| \equiv$const and either $|du| \equiv 0$ or $a \equiv 0$, i.e., $Ric \geq 0$. Suppose $0 \neq |du| \in L^2$. Then vol$M < +\infty$ and $Ric \geq 0$. Contradiction. ■

Th. 7 (Bérard, P.-Veronelli) *Let* M *be complete, let* $0 \leq \psi$ *be a* Lip_{loc} *solution of*

$$\psi \Delta \psi + a(x)\psi^2 + A|\nabla \psi|^2 \geq 0,$$

with $0 \leq a(x) \in C^0(M)$, $A \in \mathbb{R}$. *Assume*

(i) $\lambda_1(\mathcal{L}_H) \geq 0$ *for some* $H > A + 1 > 0$, $\mathcal{L}_H = -\Delta - Ha(x)$.

(ii) $\int_{B_R} \psi^{2p} = o(R^2)$, *for some* $p_0 < p < p_1$, *where* $p_0 \leq H \leq p_1$ *roots of*

$$q^2 - 2Hq + H(A+1) = 0.$$

Then $\psi \equiv$*const. and either* $\psi \equiv 0$ *or* $a \equiv 0$.

Proof. As in the usual subharmonic case, get an L^{2p}-Caccioppoli inequality for the solution ψ of $\psi\Delta\psi + a(x)\psi^2 + A|\nabla\psi|^2 \geq 0$.

Multiply both sides of the PDE by $\rho^2\psi^{2p-2}$, $\rho \in Lip_C$, and integrate by parts. The integral containing the linear term $a(x)\psi^2$ is dealt by pluggin the test function $\varphi = \rho\psi^p$ in the spectral assumption. Elaborating with the aid of Schwarz and Young inequalities gives

$$\alpha \int_M |\nabla\psi|^2 \rho^2\psi^{2p-2} \leq \beta \int_M \psi^{2p} |\nabla\rho|^2,$$

with $\alpha = -p^2 + 2Hp - H(A+1) - \varepsilon$, $0 < \varepsilon << 1$, $\beta = \beta(\varepsilon) > 0$.

By assumption $\alpha > 0$ if $0 < \varepsilon << 1$. Standard choice of ρ over balls B_R, $R \nearrow +\infty$, gives $\psi \equiv$const. If $\psi \not\equiv 0$ then vol$B_R = o\left(R^2\right)$. Combining with $\lambda_1(\mathcal{L}_H) \geq 0$ and $a \geq 0$ yields $a \equiv 0$. ■

Digression: a classical application by Schoen-Yau and some consequences.

Th. 8 *Let M^m be complete, non-cmpt. Assume $Ric_M \geq -a(x)$, with $a(x) \geq 0$ and $\lambda_1(\mathcal{L}_H) \geq 0$ for some $H > (m-1)/m$. If $D \subset\subset M$ is a domain with smooth, simply connected boundary, then there is no non-trivial homomorphism of $\pi_1(D)$ into the fundamental group of a cmpt manifold with non-positive sectional curvature.*

Proof. Take $\rho : \pi_1(D) \to \pi_1(N)$, N cmpt with $Sec_N \leq 0$. Without loss of generality we can assume $\rho = f_{\#}$ with $f : D \to N$. Since $\pi_1(\partial D) = 1$, f is homot. to const. on ∂D. We extend f to $D' \supset\supset D$ so that $f \equiv$ **const** on $\partial D'$. Extend further $f \equiv$ **const.** on $M \backslash D'$. Clearly, $|df| \in L^2$. Now apply the vanishing and conclude. ∎

Ex. 1 An extension problem. Let D be a compact manifold with $Ric_D \geq 0$ and e.g. $\partial D \approx \mathbb{S}^{m-1}$. Clearly, we can always obtain a complete, *non-compact* manifold $D \subset M$ with $Ric_M \geq -c$ by gluing (and smoothing) $\overline{\mathbb{R}^m \backslash \mathbb{B}_1(0)}$. But, in general, $(Ric_M)_-$ near ∂D cannot be too much small. Namely, if $Ric_M \geq -a(x)$ then the quantitative control $\lambda_1(\mathcal{L}_H) \geq 0$ for some $H > (m-1)/m$ (e.g. $Ric_M \geq 0$) could imply M^m compact.

Take any compact flat manifold $N = \mathbb{R}^m/\Gamma$, $m \geq 3$, and let $D = N \backslash \mathbb{B}_\varepsilon$, $0 < \varepsilon << 1$. Assume $D \subset M$ with $Ric_M \geq -a(x)$ and $\lambda_1(\mathcal{L}_H) \geq 0$. Since $m \geq 3$ then, by S.-V.K., the inclusion $i : D \hookrightarrow N$ induces an isomorphism $\rho = i_\# : \pi_1(D) \to \pi_1(N) = \Gamma$. Since $Sec_N = 0$ and $\partial D \approx \mathbb{S}^{m-1}$ is simply connected we can apply the Thm and deduce that M must be compact for, otherwise, $\rho \equiv 1$.

Rmk 2 Actually, if we insist that the extension M is Ricci-flat then $M = \mathbb{R}^m/\Gamma$. Indeed, since $\pi_1(M, D) = 1$, this follows from unique continuation arguments and Bieberbach theorem.

In order to extend the topological considerations to maps $f : M \to N$ with higher energies $|df| \in L^p$, $p \geq 2$, it is natural to use p-harmonic representatives.

Th. 9 (Nakauchi, P.-Veronelli) *Let $u : M \to N$ be a C^1 p-harmonic map, $p \geq 2$, with $|du| \in L^q$. Assume N complete with $Sec_N \leq 0$ and M complete, $Ric_M \geq -a(x)$ with $\lambda_1(\mathcal{L}_H) \geq 0$ for some $H > q^2/4(q-1)$. Then $u \equiv$ const. In particular, every $f : M \to N$ with $|df| \in L^p$ is homotopic to a constant.*

Proof. Again we start with a Bochner-type inequality

$$|dw|\,\Delta\,|dw| + a(x)\,|dw|^2 \geq -\langle dw, d\Delta w\rangle\,,\; w \in C^\infty.$$

Since u is not smooth and $\Delta_p\,|du|$ is degenerate where $|du| = 0$, we use a procedure by Duzaar-Fuchs: C^1-approximate u on $M_+ = \{|du| > 0\}$ by smooth u_k (not p-harmonic). Prove an L^q-Caccioppoli type inequality for $|du_k|$. The Caccioppoli contains an extra term that vanishes as $k \to +\infty$. Take limits to get a Caccioppoli for $|du|$ on M_+:

$$A\int_{M_+} \xi^2\,|du|^{q-2}\,|\nabla\,|du||^2 \leq B\int_{M_+} |du|^q\,|\nabla\xi|^2\,, \quad \forall \xi \in W_0^{1,2}(M_+).$$

To extend it from M_+ to M choose $\xi = \varphi_\varepsilon \eta$, with $0 \leq \eta \in C_c^\infty(M)$ and

$$\varphi_\varepsilon = \min\left\{\frac{|du|^{q/2}}{\varepsilon}, 1\right\}.$$

Finally, take limits as $\varepsilon \to 0$. ■

II.c. Comparisons of p-harmonic maps

As for general comparisons alluded to in (B) we have the following classical

Th. 10 (Schoen-Yau) *Let $u, v : M \to N$ be homotopic harmonic maps with $|du|^2 + |dv|^2 \in L^1$. If $\mathrm{vol}\,(M) < +\infty$ and $Sec_N < 0$ then, either $u = v$ or $u\,(M)\,, v\,(M) \subset \Gamma$ geodesic of N.*

Proof. Focus on some key points. Lift u, v to π_1-equivariant harmonic maps $u', v' : M' \to N'$ between universal coverings (π_1 acts by isometries). Then $(u', v') : M^{'} \to N' \times N'$ is (equivariant) harmonic. Define

$$\rho\,(x) = \mathrm{dist}_{N'} \circ (u'(x'), v'(x')) : M \to \mathbb{R}_{\geq 0}$$

where x' is any point in the fiber over x. Since N' is Cartan-Hadamard then $\mathrm{dist}_{N'}$ is convex. Harmonic maps pull harmonic functs back to subharm functs. Therefore $\Delta\rho \geq 0$. Consider $h = \sqrt{1 + \rho^2}$. Then, $\Delta h \geq 0$. Moreover, $|du|^2 + |dv|^2 \in L^1 \Longrightarrow |\nabla h|^2 \in L^1$. Now use a Liouville-type theorem to deduce $h \equiv \mathrm{const}$. This implies $\rho \equiv \mathrm{const}$. Etc... ■

The project is to extend the comparison theory to p-harmonic maps. Note that Schoen-Yau arguments does not work in this general setting due to the (nonlinear) structure of the p-Laplace operator Δ_p. A basic obstruction is that u and v p-harmonic$\not\Rightarrow$ (u, v) p-harmonic. More importantly, we have the following

Th. 11 (Veronelli) *There exist Riemannian manifolds M, N, a convex function $H : N \to \mathbb{R}$ and a p-harmonic map $u : M \to N$, for some $p > 2$, such that $H \circ u : M \to \mathbb{R}$ is not p-subharmonic.*

II.d. New comparisons for finite-energy p-harmonic maps

We need to record some facts from potential theory. Let $1 < p < +\infty$.

Def. 2 M *is p-parabolic if* $\Delta_p u \geq 0$, $\sup_M u < +\infty \Rightarrow u \equiv$ const.

There are a number of equivalent definitions of parabolicity. The first one is classical and involves the concept of capacity.

Th. 12 M *is* p*-parabolic* $\Longleftrightarrow \forall K \subset\subset M$,

$$\mathrm{cap}_p(K) = \inf \int_M |\nabla \varphi|^p = 0,$$

the infimum being taken with respect to all $\varphi \in C_c^\infty$ *such that* $\varphi \geq 1$ *on* K.

Interpretation: every $K \subset\subset M$ has a small mass from the viewpoint of p-harmonic functions.

The next result is known as the Kelvin-Nevanlinna-Royden criterion (KNR for short). It is due to T. Lyons and D. Sullivan ($p = 2$) and V. Gol'dshtein and M. Troyanov ($p > 1$).

Th. 13 *M is p-parabolic* $\Longleftrightarrow \forall X \in L^{\frac{p}{p-1}}$ *vector field s.t.* $(\operatorname{div} X)_- \in L^1$,

$$\int_M \operatorname{div} X = 0.$$

Interpretation: from the viewpoint of X, the "boundary" of M is negligible (or X has zero "boundary values"). Therefore, a global version of Stokes theorem holds. In a sense, the celebrated Gaffney(-Karp) version of Stokes theorem is in the same spirit: take $p = +\infty$ and $X \in L^1$. Here ∞-parabolicity = geodesic completeness (thanks to Troyanov for this remark).

Proof (of $\Rightarrow$). Let $\Omega_j \subset\subset M$ be s.t. $\Omega_j \nearrow M$. Since

$$\mathrm{cap}_p(\Omega_1) = 0,$$

we can choose $0 \leq \varphi_j \in C_c^\infty(\Omega_j)$ s.t.

$$\varphi_j = 1 \text{ on } \Omega_1, \text{ and } \left\|\nabla\varphi_j\right\|_{L^p} \to 0.$$

Apply Stokes theorem

$$0 = \int_M \mathrm{div}\left(X\varphi_j\right) = \int_M \varphi_j \,\mathrm{div}\, X + \int_M \left\langle X, \nabla\varphi_j\right\rangle.$$

To conclude, note that

$$\left|\int_M \left\langle X, \nabla\varphi_j\right\rangle\right| \leq \|X\|_{L^{\frac{p}{p-1}}} \|\nabla\varphi\|_{L^p} \to 0$$

and

$$\int_M \varphi_j \,\mathrm{div}\, X \to \int_M \mathrm{div}\, X.$$

■

Geometric conditions implying p-parabolicity rely on volume growth properties.

Th. 14 *Let* $(M, \langle , \rangle)$ *be complete. Consider the following growth conditions:*

(i) $\mathrm{vol}\,(B_R)^{\frac{1}{p-1}} = O\left(R^{1+\frac{1}{p-1}} \log R \log^{(2)} R \cdots \log^{(k)} R\right)$, *as* $R \to +\infty$.

(ii) $\displaystyle\int^{+\infty} \frac{R^{\frac{1}{p-1}}}{\mathrm{vol}(B_R)^{\frac{1}{p-1}}} dR = +\infty.$

(iii) $\displaystyle\int^{+\infty} \frac{dR}{\mathrm{area}(\partial B_R)^{\frac{1}{p-1}}} = +\infty.$

Then, $\quad (i) \underset{\not\Longleftarrow}{\Rightarrow} (ii) \underset{\not\Longleftarrow}{\Rightarrow} (iii) \underset{\not\Longleftarrow}{\Rightarrow} M$ *is* p*-parabolic.*

Ex. 2 *(recall Schoen-Yau Th.)* $\mathrm{vol}\,(M) < +\infty \Rightarrow p$*-parabolicity,* $\forall p > 1$.

Here is our new global comparison for vector-valued maps.

Th. 15 (Holopainen-P.-Veronelli) *Let $u, v : M \to \mathbb{R}^n$ satisfy*

$$\Delta_p u = \Delta_p v$$

and $|du| + |dv| \in L^p$, for some $p > 1$. If M is p-parabolic then $u - v \equiv$ **const.**

Proof (idea). Set $u(x_0) = v(x_0) = 0 \in \mathbb{R}^m$ and $\forall A > 0$, let

$$X_A := \left[dh_A|_{(u-v)} \circ \left(|du|^{p-2} du - |dv|^{p-2} dv \right) \right]^\sharp,$$

where $h_A(y) := \sqrt{A + |y|^2}$. Apply the KNR criterion to deduce

$$\int_M \operatorname{div} X_A = 0.$$

Take the limit as $A \to +\infty$ and conclude

$$0 = \int_M |du - dv|^p .$$

■

Note that $\mathbb{R}^n$ is contractible, hence u, v are homotopic. Therefore if u, v are p-harmonic, the previous result follows from the next

Th. 16 (P.-Rigoli-Setti) *Let $u : M \to N$ be a p-harmonic map with $|du| \in L^p$, $p > 2$. Assume that M is p-parabolic and $Sec_N \leq 0$. If u is homotopic to a constant then $u \equiv$ const.*

Very recently, the complete analogue of Schoen-Yau comparison has been finally obtained.

Th. 17 (Veronelli) *Let $u, v : M \to N$ be C^1, homotopic, p-harmonic maps with $|du|^p + |dv|^p \in L^1$. If M is p-parabolic and $Sec_N < 0$ then, either $u = v$ or $u(M), v(M) \subset \Gamma$ geodesic of N.*

III. Sobolev inequalities and p-Laplacian

Say that $(M^m, \langle , \rangle)$ enjoys an $L^{p^*,p}$-Sobolev inequality, $1/p - 1/p^* = 1/m$, if

$$\|\varphi\|_{L^{p^*}} \leq S_p \|\nabla\varphi\|_{L^p}, \tag{SI$_p$}$$

$\forall\varphi \in C_c^\infty$ and for some constant $S_p > 0$.

Rmk 3 *If M is complete with* vol $(B_R) \leq CR^m$ *then, by density arguments, (SI$_p$) extends to $\varphi \in L^{p^*}$ satisfying $|\nabla\varphi| \in L^p$.*

In $\mathbb{R}^m$ inequality (SI$_p$) holds and the explicit value of the optimal Sobolev constant K_p is known. In general, $K_p \leq S_p$ and the validity of (SI$_p$) (especially when combined with curvature conditions) introduces a number of constraints on the geometry and the topology of M. Let us consider some examples.

III.a. Rigidity under Sobolev inequalities

Th. 18 (Carron, Akutagawa, Saloff-Coste) *Assume the validity of (SI$_p$). Then* $\exists\gamma > 0$ *s.t.* $\mathrm{vol}\,(B_R) \geq \gamma\mathrm{vol}\,(\mathbb{B}_R)$, *where* $\mathbb{B}_R \subset \mathbb{R}^m$.

Th. 19 (Anderson, Li) *Assume the validity of (SI$_p$) and* $\mathrm{vol}(B'_R) \lesssim \mathrm{vol}\,(\mathbb{B}_R)$ *where* $B'_R \subset M'$, M'*=the universal covering of* M *(e.g.* $Ric_M \geq 0$*). Then* $|\pi_1(M)| < +\infty$.

Th. 20 (Ledoux, Xia) *Let* $Ric_M \geq 0$ *and assume the validity of (SI$_p$). If* S_p *is sufficiently close to* K_p *then* M *is diffeomorphic to* $\mathbb{R}^m$. *If* $S_p = K_p$ *then* M *is isometric to* $\mathbb{R}^m$.

Proof (P.-Veronelli). Crucial point: use the curvature condition to improve lower volume estimate. Recall that, in $\mathbb{R}^m$, the equality in (SI_p) is realized by the (radial) Aubin-Talenti functions

$$\varphi_\lambda(|x|) = \frac{\beta(m,p)\,\lambda^{\frac{m-p}{p^2}}}{\left(\lambda + |x|^{\frac{p}{p-1}}\right)^{\frac{m}{p}-1}}.$$

which satisfy

$$\int_{\mathbb{R}^m} \varphi_\lambda^{p^*} = 1,$$

and obey the nonlinear Yamabe equation

$${}^{\mathbb{R}^m}\Delta_p \varphi_\lambda = -K_p^{-p} \varphi_\lambda^{p^*-1}.$$

Define $\widehat{\varphi}_\lambda : M \to \mathbb{R}$ as $\widehat{\varphi}_\lambda(x) := \varphi_\lambda(r(x))$ and consider the vector field

$$X_\lambda := \widehat{\varphi}_\lambda\,|\nabla\widehat{\varphi}_\lambda|^{p-2}\,\nabla\widehat{\varphi}_\lambda.$$

Then, by volume comparison, $X_\lambda \in L^1(M)$. Also, by Laplacian comparison,

$$\Delta_p \widehat{\varphi} \geq -K_p^{-p} \widehat{\varphi}_\lambda^{p^*-1}.$$

Therefore,

$$\operatorname{div} X_\lambda \geq \widehat{\varphi}_\lambda \Delta_p \widehat{\varphi}_\lambda \geq -K_p^{-p} \widehat{\varphi}_\lambda^{p^*} \in L^1(M).$$

Using the Karp version of Stokes theorem we deduce

$$0 = \int_M \operatorname{div} X_\lambda \geq \int_M |\nabla \widehat{\varphi}_\lambda|^p - K_p^{-p} \int_M \widehat{\varphi}_\lambda^{p^*},$$

that is

$$\frac{\int_M |\nabla \widehat{\varphi}_\lambda|^p}{\int_M \widehat{\varphi}_\lambda^{p^*}} \leq K_p^{-p}. \tag{*}$$

On the other hand, using $\widehat{\varphi}_\lambda$ in (SI_p) we obtain

$$S_p^{-p} \leq \frac{\int_M |\nabla \widehat{\varphi}_\lambda|^p}{\left(\int_M \widehat{\varphi}_\lambda^{p^*}\right)^{\frac{p}{p^*}}}. \tag{**}$$

Take the quotient (**)/(*):

$$\int_{\mathbb{R}^m} \varphi_\lambda^{p^*} = 1 \leq \int_M \left(\frac{S_p}{K_p}\right)^m \widehat{\varphi}_\lambda^{p^*}.$$

Integrating by parts in polar-coordinates,

$$0 \leq \int_0^\infty \left[\left(\frac{S_p}{K_p}\right)^m \frac{V(B_t)}{V(\mathbb{B}_t)} - 1\right] V(\mathbb{B}_t) \frac{d}{dt}\left(-\varphi_\lambda^{p^*}(t)\right) dt$$

where, by Bishop-Gromov, $V(B_t)/V(\mathbb{B}_t) \searrow$. Since $\frac{d}{dt}(-\varphi_\lambda^{p^*}(t)) \searrow 0$ uniformly on compact intervals, as $\lambda \to +\infty$, then the term in $[\cdots]$ is ≥ 0, i.e.:

$$\mathrm{vol}\,(B_R) \geq (K_p/S_p)^m \,\mathrm{vol}\,(\mathbb{B}_R)\,,\ \forall R > 0.$$

The desired rigidity now follows either from the equality case in Bishop-Gromov $(K_p/S_p = 1)$ or from Cheeger-Colding theory $(K_p/S_p \approx 1)$ ■

Rmk 4 (P.-Veronelli) A similar proof works if we replace $Ric_M \geq 0$ with the asymptotic condition $Ric_M \geq -G(r(x))$ where $r(x) = d(x,o)$, $o \in M$ is a reference origin, and $G \geq 0$ satisfies

$$\int_0^{+\infty} tG(t)\,dt = b_0 < +\infty.$$

The corresponding rigidity (diffeomorphic rigidity) holds under the curvature requirement $Sec_M \geq -G(r(x))$ when b_0 is sufficiently close to 0.

Rmk 5 (Carron) Last July, Carron generalized our volume estimate by replacing the asymptotic curvature condition with an asymptotic volume condition. Namely

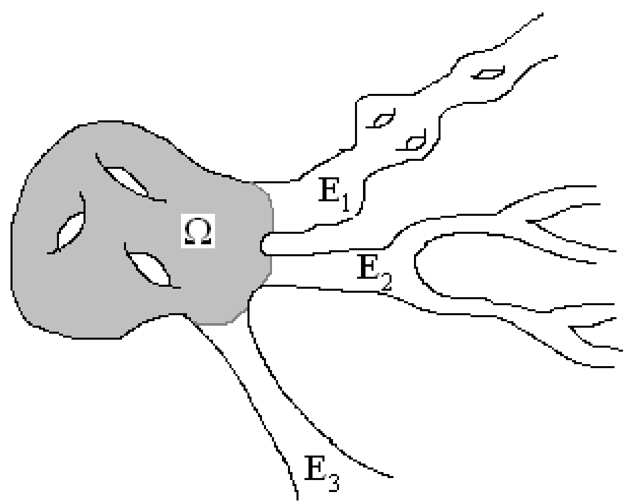
$$\beta := \lim_{R\to+\infty} \frac{\mathrm{vol}(B_R)}{\mathrm{vol}(\mathbb{B}_R)} < +\infty \ + \ (SI_p) \Rightarrow \beta \geq (K_p/S_p)^m$$

The proof is a simple but very clever elaboration of the original argument by Ledoux. Chapeau!

III.b. Sobolev inequalities and topology at infinity

In the presence of the Sobolev inequality (SI_p) we are able to bulid a link between the analysis of p-harmonic functions and the topology at infinity of the underlying complete, non-compact manifold M.

Def. 3 *An end E of M with respect to $\Omega \subset\subset M$ is any of the unbounded connected components of $M \backslash \Omega$. Say that M is connected at infinity if, for evey smooth $\Omega \subset\subset M$, $M \backslash \Omega$ has exactly one end.*



Ex. 3 M =universal covering of a cmpt manifold N with $\pi_1(N) = \mathbb{Z}^{k\geq 2}$. Then M is connected at infinity. Indeed, by Švarc-Milnor theory, M is quasi-isometric to the Cayley graph G of $\pi_1(N)$. The number of ends is a quasi-isometry invariant+G connected at infinity$\Rightarrow$ M connected at infinity.

Ex. 4 $M = N \times \mathbb{R}$ with N cmpt is disconnected at infinity.

Ex. 5 $M = N \times \mathbb{R}^k$, with $k \geq 2$, is connected at infinity.

Ex. 6 (Cheeger-Gromoll) Assume $Ric_M \geq 0$ and $Ric_M(x) > 0$ for some $x \in M$. Then M is connected at infinity. Indeed, if $M \backslash \Omega$ has two unbounded components E_1, E_2 then M contains a line. Since $Ric_M \geq 0$ we have isometric splitting $M = N \times \mathbb{R}$. This violates the assumption $Ric_M(x) > 0$ somewhere.

In the presence of a general $L^{q,p}$-Sobolev inequality the curvature assumption in Cheeger-Gromoll Ex. 6 can be considerably relaxed.

Th. 21 *Let $(M^m, \langle , \rangle)$ be a complete manifold satisfying the Sobolev inequality*

$$\|\varphi\|_{L^q} \leq S \|\nabla \varphi\|_{L^p},$$

for some $S > 0$ and $1/p - 1/q \leq 1/m$. Assume that $Ric \geq -a(x)$ where $a(x) \geq 0$ is small in the spectral sense

$$\lambda_1 (-\Delta - Ha(x)) \geq 0,$$

for some $H > p^2/4(p-1)$ if $p > 2$, and $H > (m-1)/m$ if $p = 2$. Then, M is connected at infinity.

It is a contribution of several people and inspires to harmonic function theory developed by P. Li, L.-F. Tam and collaborators. In case $p = 2$, versions of this result are due to P. Li and J. Wang, H.-D. Cao, Y. Shen, S. Zhu. The general case $p \neq 2$ was observed e.g. by P.-Setti-Troyanov. The proof is in three steps.

(a) Sobolev inequality (SI_p) $\Rightarrow$ every end E has infinite volume and is "large" in the sense of potential theory, i.e., E is p-hyperbolic=not p-parabolic. (Bukley-Koskela, P.-Setti-Troyanov)

(b) If M has two p-hyperbolic ends, construct a non-constant p-harmonic function $u \in C^1(M)$ satisfying $|\nabla u| \in L^p$. (Holopainen, P.-Setti-Troyanov)

(c) Curvature assumption + corresponding vanishing result $\Rightarrow$ $u \equiv$ **const.** (this has been already discussed)

(a) volume and potential theory of ends. Basic idea: since

$$\|\varphi\|_{L^q} \leq S \|\nabla\varphi\|_{L^p},$$

if we fix $K \subset\subset M$ and choose $\varphi = 1$ on K, then

$$\|\nabla\varphi\|_{L^p} \geq S^{-1}\mathrm{vol}\,(K)^{1/q}.$$

This means

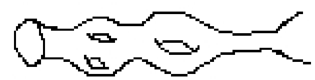
$$\mathrm{cap}_p\,(K) \geq S^{-1}\mathrm{vol}\,(K)^{1/q} > 0$$

and the manifold is p-hyperbolic. Also, by Carron-Akutagawa volume estimates,

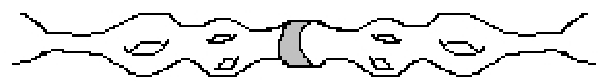
$$\mathrm{vol}\,(B_R) \geq CR^m \to +\infty.$$

All these considerations can be localized on each end.

Def. 4 *Say that the end E of M is p-parabolic if its Riemannian double $\mathcal{D}(E)$ is p-parbolic as a manifold without boundary.*



E



D(E)

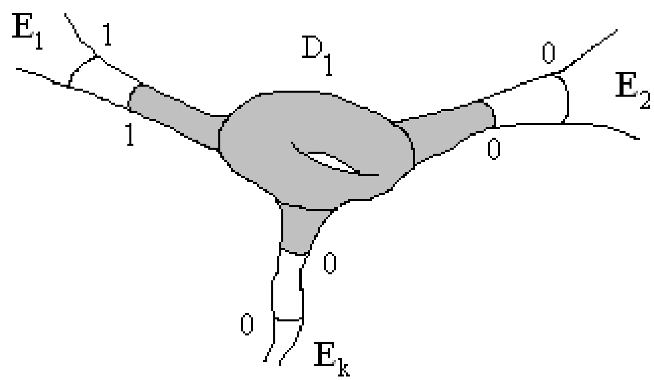
The key point to localize the above arguments on E is the next

Th. 22 (Carron, P.-Setti-Troyanov) *The $L^{q,p}$ Sobolev inequality holds off a compact set if and only if it holds (with a different constant) on all of M*

(b) construction of the p-harmonic function

Let E_1, E_2,...,E_k be the ends of M, $k \geq 2$. By (a) they are p-hyperbolic. Take an exhaustion $D_j \nearrow M$. For every j solve the Dirichlet problem

$$\begin{cases} \Delta_p u_j = 0 & \text{on } D_j \\ u_j = 0 & \text{on } E_1 \cap \partial D_j \\ u_j = 1 & \text{on } (M \backslash E_1) \cap \partial D_j. \end{cases}$$



By the maximum principle $u_j \nearrow$ and, therefore, we can define

$$u(x) = \lim_j u_j(x).$$

Then:

1) u is p-harmonic by the Harnack principle.

2) Using the fact that there are at least two p-hyperbolic ends it can be shown that u is nonconstant.

3) Using capacitary arguments it follows $\left\|\nabla u_j\right\|_{L^p} \leq C$, $\forall j$. This implies $|\nabla u| \in L^p$.

This completes the proof of the Theorem.