Geometric aspects of recurrence, non-explosion, and Feller property of a Riemannian manifold

Stefano Pigola
Università dell'Insubria, Como

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The heat kernel of (M, g) is the smooth, minimal, positive fundamental solution of the heat equation on $M \times M \times (0, +\infty)$:

$$\begin{cases} \Delta_x p\,(x, y, t) = \partial_t p\,(x, y, t) \\ p(x, y, 0^+) = \delta\,(x, y) \end{cases}$$

Existence of $p\,(x, y, t)$ was proved e.g. by J. Dodziuk. Basic properties:

- (symmetry) $p\,(x, y, t) = p\,(y, x, t) > 0$
- (semigroup property) $p\,(x, y, t) = \int_M p\,(x, z, t-s)\, p\,(z, y, s)\, dz$
- (sub-Markovianity) $\int_M p\,(x, y, t)\, dy \leq 1$

Associated to $p(x,y,t)$ there is a stochastic process X_t with continuous trajectories, the Brownian motion of (M,g). The probability that the Brownian particle X_t starting at x lies in A at time t is given by:

$$\mathbb{P}_x(X_t \in A) = \int_A p(x,y,t)\,dy.$$

The global behaviour of the heat kernel, both in time and in space, reflects special properties of the Brownian paths: they are typically called recurrence (or parabolicity), non-explosion (or stochastic completeness) and Feller property (or C_0-diffusion property).

We are going to take a tour into them from the purely deterministic viewpoint of PDEs. We will allude to the stochastic aspects of the theory only to make the picture more intuitive.

A. Recurrence

In some sense, recurrence is a kind of compactness. Replace geodesics with Brownian paths. On a cmpt Riemannian manifold geodesics wrap indefinitely on the space. On a recurrent manifold, Brownian paths enter infinitely many times a fixed compact domain. This analogy tourns out to be visible from the deterministic viewpoint.

Def. 1 *Say that X_t on (M,g) (or (M,g) itself) is recurrent if, for every domain $D \subset\subset M$,*

$$\mathbb{P}_x\left(\exists\,\{T_n\} \nearrow +\infty : X_{T_n} \in D\right) = 1.$$

From the heat kernel viewpoint this property is related to a global time-integrability.

Def. 2 *Say that* (M, g) *is* *non-parabolic* *if*

$$G(x,y)=\int_0^{+\infty} p(x,y,t)\,dt$$

is a genuine function on $M \times M \backslash \Delta$*, the* *Green kernel* *of* M*. Otherwise,* M *is* *parabolic*.

The Green kernel (if any) $G(x,y)=G(y,x)>0$ is the minimal, positive fundamental solution of the Laplace equation:

$$\Delta_x G(x,y)=-\delta_y(x).$$

It has integrable singularities along the diagonal $\Delta \subset M \times M$ with

$$G(x,y) \sim C_m \begin{cases} -\log d(x,y) & m=2 \\ d(x,y)^{2-m} & m \geq 3, \end{cases}$$

as $d(x,y) \to 0$.

A.1. Equivalent formulations

Th. 3 *The following properties are equivalent to recurrence of* (M, g)*:*

- *(Green kernel)* (M, g) *is parabolic.*
- *(Kha'sminskii test)* $\exists u > 0$, $u \to +\infty$ *such that* $\Delta u \leq 0$ *off a cmpt set.*
- *(Liouville) If* $\Delta u \geq 0$ *and* $\sup_M u < +\infty$ *then* $u \equiv \text{const}$.
- *(Ahlfors max princ)* $\forall \Omega$ *domain,* $\forall u$ *s.t.* $\Delta u \geq 0$ & $\sup_\Omega u < +\infty$,
$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$
- *(Stokes thm)* $\forall X \in L^2(M)$ *with* $(\operatorname{div} X)_- \in L^1(M)$ *it holds*
$$\int_M \operatorname{div} X = 0.$$

A.2. Geometric conditions

Parabolicity of a complete manifold (M, g) relies on quadratic volume growth properties of the space. Let $B_R(p)$ and $\partial B_R(p)$ be the metric balls and spheres of M.

Th. 4 *A complete (M, g) is parabolic if, for any reference origin $p \in M$,*

$$\int^{+\infty} \frac{dR}{|\partial B_R(p)|} = +\infty.$$

Rmk 5 *Note the chain of implications*

$$|B_R(p)| = O(R^2) \Rightarrow \int^{+\infty} \frac{R}{|B_R(p)|} dR = +\infty \Rightarrow \int^{+\infty} \frac{dR}{|\partial B_R(p)|} = +\infty.$$

A.3 Manifolds with boundary

Let (M, g) be a complete manifold with $\partial M \neq \emptyset$ and outer normal ν. What is recurrence in this setting? At least two possibilities:

$\mathcal{D}$-recurrence. Dirichlet heat Kernel = absorbed Brownian motion hits ∂M (and dies) in finite time

$\mathcal{N}$-recurrence. Neumann heat kernel = reflected Brownian motion enters infinitely many times a compact domain.

The following are classical characterizations in terms of Liouville properties. They can be found in papers by minimal surfaces people and Grigor'yan.

Th. 6 M *is* $\mathcal{D}$*-recurrent* $\Longleftrightarrow$ *$\mathcal{D}$-Liouville:*

$$\begin{cases} \Delta u = 0, \text{ in } \mathrm{int} M \\ u = 0, \text{ on } \partial M \\ \sup_M |u| < +\infty \end{cases} \Rightarrow u \equiv 0.$$

Th. 7 M *is* $\mathcal{N}$*-recurrent* $\Longleftrightarrow$ *$\mathcal{N}$-Liouville:*

$$\begin{cases} \Delta u \geq 0, \text{ in } \mathrm{int} M \\ \partial_\nu u \leq 0, \text{ on } \partial M \\ \sup_M u < +\infty \end{cases} \Rightarrow u \equiv \mathrm{const}.$$

Moreover, if (M, g) *is complete and* $|B_R| = O(R^2) \Rightarrow \mathcal{N}$*-recurrence*

The following version of the Ahlfors max principle is useful in applications.

Prop. 8 (Impera-P.-Setti, '13) *Let M be $\mathcal{N}$-recurrent. Then:*

$$\begin{cases} \Delta u \geq 0, \text{ in } \mathrm{int} M \\ \sup_M u < +\infty \end{cases} \Rightarrow \sup_M u = \sup_{\partial M} u.$$

In particular:

- *$\mathcal{N}$-recurrence $\Rightarrow$ $\mathcal{D}$-recurrence.*

- *Hence, (M, g) complete and $|B_R(p)| = O(R^2) \Rightarrow \mathcal{D}$-recurrence.*

A.4. Height estimate for H-graphs

Let (N, h) be a complete, n-dimensional manifold, $\partial N = \emptyset$.

Let $\mathbf{\Sigma} \hookrightarrow N \times \mathbb{R}$ be the proper graphical surface

$$\mathbf{\Sigma} = Graph_{\Omega}(f) = \{(x, f(x)) : x \in \Omega\}$$

with boundary

$$\partial\mathbf{\Sigma} = Graph_{\partial\Omega}(f)$$

where $\overline{\Omega} \subseteq N$ a closed domain, $\partial\Omega \in C^{\infty}$, and $f \in C^{\infty}(\Omega) \cap C^{0}(\overline{\Omega})$. Orient $\mathbf{\Sigma}$ by the downward pointing Gauss map

$$\mathcal{N}(x) = \frac{(\nabla_N f(x), -1)}{\sqrt{1 + |\nabla_N f(x)|^2}}.$$

The mean curvature of $\mathbf{\Sigma}$ w.r.t. $\mathcal{N}$ is given by

$$H(x) = -\frac{1}{n}\,\mathsf{div}_N \left\{ \frac{\nabla_N f(x)}{\sqrt{1 + |\nabla_N f(x)|^2}} \right\}.$$

Say that $\mathbf{\Sigma}$ is an $H\,(> 0)$-graph if $H(x) \equiv H > 0$ is constant.

Th. 9 (Impera-P.-Setti, '13) *Let N^2 be a complete surface with Gauss curvature $\mathsf{K}_N \geq 0$ and $\partial N = \emptyset$. Let $\overline{\Omega} \subset N$ be a closed domain with smooth boundary $\partial\Omega \neq \emptyset$. If $\mathbf{\Sigma} = Graph_{\Omega}(f)$ is a proper $H\,(> 0)$-graph with $\partial\mathbf{\Sigma} \subset N \times 0$ then*

$$\mathbf{\Sigma} \subseteq N \times [0, 1/H].$$

Parabolicity $\approx$ compactness $\Rightarrow$ we follow Rosenberg approach in the cmpt case. Two ingredients:

(i) A-priori boundedness of CMC graphs

Th. 10 (Lopez-Ros '89, Elbert-Nelli-Rosenberg '07) *Let* $\bar{\Omega} \subseteq N^n$, *a complete Riemannian manifold with* $\mathrm{Sec}_N \geq 0$ *and* $n \leq 4$. *Given* $H > 0$, $\exists C = C(H,n) > 0$ *s.t. if* $\Sigma = Graph_\Omega(f)$ *is an* H*-graph on* $\Omega \subseteq N$ *with* $\partial\Sigma \subset N \times 0$, *then* $\Sigma \subset N \times [-C, C]$. *(dim* ≤ 4 *technical or substantial?)*

(ii) $\Sigma \hookrightarrow N \times \mathbb{R}$ inherits the volume growth of N

Th. 11 (Li-Wang '01, Impera-P.-Setti) *Let* $\Sigma = Graph_\Omega(f)$ *with mean curvature* $H(x)$. *If* $\|H \cdot f\|_\infty < +\infty$ *and* $f = 0$ *on* $\partial\Sigma$, *then the intrinsic volume of* Σ*-balls is controlled by that of* N*-balls*

$$\left|B_R^\Sigma\right| \leq C \left|B_{2R}^N \cap \Omega\right|. \text{ (}N\text{ recurrent} \Rightarrow \Sigma \text{ recurrent?)}$$

Rmk 12 *We can assume $f \in C^{\infty}(\overline{\Omega})$. The general case follows by considering the smooth super-graph $\Sigma_{\varepsilon} = Graph_{\overline{\Omega}_{\varepsilon}}(f_{\varepsilon})$ where, for $0 < \varepsilon << 1$,*

$$\overline{\Omega}_{\varepsilon} = \{\varepsilon \leq f(x) < +\infty\}, \qquad f_{\varepsilon}(x) = f(x) - \varepsilon.$$

We get

$$f_{\varepsilon} \leq \frac{1}{H} \text{ i.e. } f \leq \frac{1}{H} + \varepsilon$$

and then $\varepsilon \to 0$. The estimate $f \geq 0$ is similar.

Proof (of $f \leq 1/H$) Since dim $N = 2$, N is complete and $K_N \geq 0$ then, by Bishop-Gromov, $\left|B_R^N\right| = O(R^2)$. Since Σ is a complete $H\,(>0)$-graph, from Th. 11 $\left|B_R^{\Sigma}\right| = O(R^2)$, hence Σ is parabolic.

Let $\Theta(x) = \widehat{\frac{\partial}{\partial t}\mathcal{N}(x)})$ the vertical angle. Then

$$\cos\Theta = -\frac{1}{\sqrt{1+|\nabla_N f|^2}} < 0.$$

The mean curv. eq. reads

$$\Delta_{\Sigma} f = nH \cos\Theta.$$

Moreover, Σ is stable and $\cos\Theta$ is a Jacobi field:

$$\Delta_{\Sigma} \cos\Theta + \|\sigma\|^2 \cos\Theta \geq 0,$$

with σ the 2^{nd} fund form. Consider the new funct

$$\boxed{\xi(x) = H \cdot f(x) + \cos\Theta(x)}$$

Combining mean curv eq + stability eq $\Rightarrow$ ξ is subharmonic:

$$\Delta_{\Sigma}\xi = \left(nH^2 - \|\sigma\|^2\right)\cos\Theta \geq 0, \text{ on } \Sigma.$$

Since $\xi \leq 0$ on $\partial\Sigma$, the Ahlfors max principle $\Rightarrow$ $\xi \leq 0$, on Σ. Thus:

$$\xi(x) \leq 0 \Rightarrow f(x) \leq -\frac{\cos\Theta}{H} \leq \frac{1}{H}. \blacksquare$$

B. (Non-)Explosion

A tool to get information on global solutions of PDEs like $\Delta u \geq f(u)$.

Sometimes recurrence (=parabolicty) is too much strong. In some sense it limits ourselves to cmpct behaviours. What about complete non-compact behaviours? (Non-)explosion can be considered as a Brownian counterpart of geodesic completeness. Like geodesics (special divergent paths), Brownian paths a.s. do not diverge to infinity in finite time.

Def. 13 *Say that (M, g) is stochastically complete (X_t does not explode) if*

$$\mathbb{P}_x(X_T = \infty, T < +\infty) = 0,$$

where $\overline{M} = M \cup \{\infty\}$ Alexandrov compactification.

B.1. Equivalent formulations

Th. 14 *The following properties are equivalent to non-explosion of* (M, g)*:*

- *(Heat semigroup)* $\int_M p(x, y, t)\, dy \equiv 1$.

- *(Kah'sminskii test)* **[Mari-Valtorta, TAMS '13]**
$\exists u(x) > 0$, $u(x) \to +\infty$ *as* $x \to \infty$ & $\Delta u \leq \lambda u$ *outside a cmpt set.*

- *(Liouville) If* $\Delta u \geq \lambda u$ *on* M & $0 \leq u \leq \sup_M u < +\infty$ *then* $u \equiv 0$.

- *(Max.Princ. at infinity)* **[P.-Rigoli-Setti, PAMS '03]**
Let $u \leq \sup_M u < +\infty$ *then there exists* $\{x_k\} \subset M$ *s.t*

(i) $u(x_k) \geq \sup_M u - 1/k$, *(ii)* $\Delta u(x_k) \leq 1/k$.

B.2 Geometric conditions

(Non-)explosion relies on quadratic-exp volume growth properties. These are implied by curvature restrictions.

Th. 15 *Let (M, g) be complete. Then M is stochastically complete provided either of the following conditions hold:*

- **[Hsu, Ann. Prob.'89]**

  $$\text{Ric} \geq -(m-1)\, G(r(x)),\ \textit{with } G \nearrow \ \&\ \int^{+\infty} G(r)^{-1/2}\, dr = +\infty.$$

- **[Grigor'yan, Soviet Math. Dokl. '87]**

  $$\int^{+\infty} \frac{R\, dR}{\log |B_R(p)|} = +\infty.\ \textit{Ex: } |B_R(p)| \leq Ce^{r^2} \Rightarrow \textit{stoch. compl.}$$

B.3. Mean curvature estimates

Classical results by L. Jorge and F. Xavier '81, now extended in various ways, give an estimate of the mean curvature of certain complete submanifolds confined in balls and cylinders. The structure of these results is as follows. Given an ambient manifold N^n:

$$\begin{cases} \varphi : \boldsymbol{\Sigma}^m \hookrightarrow N^n \text{ isometric immersion} \\ \boldsymbol{\Sigma} \text{ satisfies some geometric assmpt} \\ \varphi(\boldsymbol{\Sigma}) \subseteq \Omega \text{ special domain of } N \end{cases} \Rightarrow \sup_{\boldsymbol{\Sigma}} |\mathbf{H}| \geq C,$$

where $\mathbf{H}$ is the mean curvature vector field of the immersion and $C > 0$ depends on m, n and the geometry of Ω. This, in particular, prevents CMC hypersurfaces to be inside certain regions of N.

We are going to consider the case where the ambient N^n is Cartan-Hadamard (complete, simply connected with $\mathrm{Sec}_N \leq 0$) and Ω is "very" unbounded.

Def. 16 *Let* $\sigma : [0, +\infty) \to N^n$ *be a ray. The corresponding* *Busemann funct* *is* $b_\sigma(x) = \lim_{t\to+\infty} b_{\sigma,t}(x)$, *where* $b_{\sigma,t}(x) = d_N(\sigma(t), x) - t$.

A *horoball* *is any sublevel set* $\mathcal{B}_{\sigma,R} = b_\sigma^{-1}((-\infty, R))$. *A* *horocylinder* *is* $\mathcal{B}_{\sigma,R} \times \mathbb{R}^\ell$.

Th. 17 (Bessa-Jorge-P.-Setti, '13) *Let* N^n *be Cartan-Hadamard with*

$$-B^2 \leq \mathrm{Sec} \leq -A^2 < 0,$$

and $\varphi : \Sigma^m \hookrightarrow N^n \times \mathbb{R}^\ell$ *be properly immersed with* $\varphi(\Sigma)$ *inside a horocylnder of* $N^n \times \mathbb{R}^\ell$, $m > \ell \geq 0$. *Then, the mean curvature of* Σ *satisfies*

$$\sup_\Sigma |\mathbf{H}| \geq \frac{m-\ell}{m} A.$$

(lower bound B of Sec really needed or technical?)

Proof. In two steps.

(i) Analysis on the Busemann fnct. N is CH $\Rightarrow b_\sigma \in C^2(N)$ (P. Eberlein). Moreover, for fixed t, let $b_{\sigma,t}(x) = r_{\sigma(t)}(x) - t$. Standard *Hessian comparison*

$$(\bigstar)\ A \operatorname{coth}(A r_{\sigma(t)}) \{g_N - db_{\sigma,t} \otimes db_{\sigma,t}\} \leq \mathsf{Hess}(b_{\sigma,t}) \leq B \operatorname{coth}(B r_{\sigma(t)}) \{g_N - db_{\sigma,t} \otimes db_{\sigma,t}\}.$$

Aim: let $t \to +\infty$ and deduce the analogous estimate for b_σ.

• For C^2 fncts, distributional and classical gradient and Hessian agree. Moreover distributional formulations are stable under limits.

• Hessian comparison$\Rightarrow \left\|\mathsf{Hess}(b_{\sigma,t})\right\|$ loc.unif. bounded. Therefore, each sequence $\nabla b_{\sigma,t_n}$ is equicontnuous. Since $\left|\nabla b_{\sigma,t_n}\right| = 1$, then $\exists \nabla b_{\sigma,t_{n'}} \to X$ and, by weak convergence, $X = \nabla b_\sigma$.

• Apply the weak formulation of $(\bigstar)$, take $t \to +\infty$ and back to the classical formulation

(ii) Application of the max princ at infinity. Let $\varphi(\mathbf{\Sigma}) \subset \mathcal{B}_{\sigma,R}$. Assume also $\sup_{\mathbf{\Sigma}} |\mathbf{H}| < +\infty$ otherwise it's trivial.

• Define

$$w(x) = \exp(A \cdot b_{\sigma}(\varphi(x))).$$

Since $\varphi(\mathbf{\Sigma}) \subset \mathcal{B}_{\sigma,R} \Leftrightarrow b_{\sigma}(\varphi(x)) < R$, then $\sup_{\mathbf{\Sigma}} w < +\infty$.

• By comparison theory for the Busemann funct

$$\mathbf{\Delta} w \geq mw \left\{ A - \sup_{\mathbf{\Sigma}} |\mathbf{H}| \right\}.$$

• $\mathbf{\Sigma}$ is stochastically complete. Indeed, by Hessian comparison, $u(x) = d_N(\varphi(x), o)$ satisfies $\mathbf{\Delta} u \leq u$ off a compact set, and by the properness assumption, $u \to +\infty$. Therefore, Kah'sminskii test applies.

• Use stoch compl. in the form of maximum principle at infinity and conclude.
■

C. Feller property

A tool to get qualitative information on solutions of $\Delta u \geq f(u)$ on ends.

Def. 18 *Say that the Brownian motion X_t on (M, g) (or (M, g) itself) has the Feller property if, for any fixed time $t_0 > 0$ and for any fixed domain $D \subset\subset M$,*

$$\mathbb{P}_x \left(X_T \in D, T < t_0 \right) \to 0, \text{ as } x \to \infty.$$

Roughly speaking, it is increasingly more difficult for X_t to visit D before t_0 as the starting point x is put far and far away. Kendall relates this property to "implosion" at $t = 0^+$ of Brownian paths on $(0, +\infty)$. Ask to Kendall...

From the heat kernel viewpoint, the Feller property is usually called C_0-diffusion and it is related to the preservation property of the space

$$C_0(M) = \left\{ u \in C^0(M) : u(x) \to 0 \text{ as } x \to \infty \right\}.$$

C.1 Equivalent formulations

Th. 19 (Azencott, Bull. Soc. Math. Fr. '74) *The following are equivalent:*

- *(probability)* X_t *has the Feller property.*

- *(C_0-diffusion)* $p(x, y, t)$ *preserves* $C_0(M)$.

- *(PDEs) Fix any* $D \subset\subset M$ *&* $\lambda > 0$ *cnst. Let* $h : M \backslash D \to \mathbb{R}_{>0}$ *be the minimal, positive solution of the exterior problem*

$$(\blacktriangle) \qquad \begin{cases} \Delta u = \lambda u & M \backslash \overline{D} \\ u = 1 & \partial D. \end{cases}$$

Then $h(x) \to 0$, *as* $x \to \infty$.

Rmk 20 *The minimal solution $h > 0$ is easily constructed on any manifold using an exhaustion procedure. Difficulty in using the PDE viewpoint: h is not a generic solution but the minimal one. There exist M's with infinitely many bounded solutions. However:*

Th. 21 (P.-Setti, JFA '12) *Assume that (M, g) is stochastically complete. Then h is the unique, bounded solution of (▲).*

Proof. In two steps.

First step. Let $u \geq 0$ be any solution of

$$\begin{cases} \Delta u \leq \lambda u & M \backslash \overline{D} \\ u \geq 1 & \partial D. \end{cases}$$

Then $h \leq u$. This is an easy application of the standard max. princ. Indeed, $h = \lim h_n$ where h_n solve the problem on $D_n \backslash D$ and $h_n = 0$ on ∂D_n, $h_n = 1$ on ∂D. Therefore, $h_n \leq u$ for every n.

Second step. Let $u \geq 0$ be a bounded solution of

$$\begin{cases} \Delta u \geq \lambda u & M \backslash \overline{D} \\ u \leq 1 & \partial D. \end{cases}$$

Then $u \leq h$. Indeed, let $\varepsilon > 0$ and note that $v_\varepsilon = \max\{u - h - \varepsilon, 0\} \geq 0$ is a bounded subsolution on $M \backslash \overline{D}$ with $v_\varepsilon = 0$ in a neighbourhood of ∂D. Extend v_ε to be 0 on D and get a function satisfying

$$\begin{cases} \Delta v_\varepsilon \geq \lambda v_\varepsilon, \text{ on } M \\ 0 \leq v_\varepsilon \leq \sup_M u < +\infty. \end{cases}$$

By stochastic completeness (Liouville property), $v_\varepsilon \equiv 0$ i.e. $u \leq h + \varepsilon$. Now let $\varepsilon \to 0$. ■

C.2 Geometric conditions

The validity of the Feller property relies on curvature conditions. *Almost nothing is known on the role of volumes.*

Th. 22 (Hsu, Ann. Prob. '89) *The complete manifold* (M, g) *is Feller provided*

$$\text{Ric} \geq -(m-1)\,G\,(r\,(x))\,,\ \textit{with } G \nearrow\ \ \&\ \int^{+\infty} G^{-1/2}\,(r)\,dr = +\infty.$$

Rmk 23 *The condition is essentially sharp. The proof is purely probabilistic. Deterministic arguments?. The best deterministic result is the following, obtained via direct heat kernel estimates (in the tradition of Yau, Dodzjuk and Schoen-Yau for* Ric $\geq -C$*)*

Th. 24 (Li-Karp, unpublished) *The complete manifold* (M, g) *is Feller provided* Ric $\geq -C(1 + r\,(x)^2)$.

Rmk 25 *The curvature condition is the same as for stoch. compl. However the two concepts are unrelated.*

Th. 26 (Azencott, P.-Setti) *Let* $g = dr \otimes dr + \sigma(r)^2 g_{\mathbb{S}^{m-1}}$ *be a rotationally symmetric metric on* $\mathbb{R}^m$. *Then* $\mathbb{R}^m_\sigma$ *is Feller* $\Longleftrightarrow$ *either of the following conditions is satisfied:*

(a) $\int^{+\infty} \frac{dr}{|\partial B_r(0)|} < +\infty$ *(transience)*

(b) $\int^{+\infty} \frac{dr}{|\partial B_r(0)|} = +\infty$ *(recurrence) &* $\int^{+\infty} \frac{|\mathbb{R}^m_\sigma \backslash B_r(0)|}{|\partial B_r(0)|} dr = +\infty$.

Cor. 27 *We have:*

- $\left|\mathbb{R}^2_\sigma\right| = +\infty \Rightarrow$ *Feller but stoch. incompl. if* $|B_R(0)| \asymp e^{R^{2+\varepsilon}}$.
- $\left|\mathbb{R}^2_\sigma\right| < +\infty \Rightarrow$ *stoch. compl. but non-Feller if* $|\partial B_r(0)| \asymp e^{-R^{2+\varepsilon}}$.

Probl. 28 *What about volumes? Some influence on the validity of Feller?*

A satisfactory answer is unknown. *This prevents e.g. the extension to Dirichlet spaces.*

- In the easy case of models $\mathbb{R}^m_\sigma$ we have a characterization showing that, apparently, large volumes help

- A (revised) conjecture by Li-Karp states that:

$$\begin{cases} M \text{ complete } \textit{with only } 1 \textit{ end} \\ |B_{r+1}(p) \backslash B_r(p)| \geq e^{-Cr^2} \end{cases} \Rightarrow M \text{ Feller}$$

C.3. Jorge-Xavier estimates on ends

Th. 29 (Bessa-P.-Setti, Revista '13) *Let* $f : \mathbf{\Sigma}^m \to \mathbb{R}^n$ *be a complete submanifold satisfying* Ric $\geq -(m-1)\,G(r)$ *with* $\int^{+\infty} G(r)^{-1/2}\,dr = +\infty$. *Let* E *be an end of* $\mathbf{\Sigma}$. *If* E *is contained in the ball* $B_R(0)$ *then, the mean curvature of* E *satisfies*

$$\sup_E |\mathbf{H}| \geq \frac{1}{R}.$$

Proof. Note that, by Hsu, $\mathbf{\Sigma}$ is both stoch. compl. and Feller.

• Consider the function $u : E \longrightarrow \mathbb{R}_{\geq 0}$ s.t. $u(x) = \| f|_E - 0\|^2$. Then

$$\Delta u \geq 2m \left(1 - R \sup_E |\mathbf{H}|\right)$$

• By contradiction, suppose $1 - R \sup_E |\mathbf{H}| = s > 0$. Then $\Delta u \geq \lambda u$ on E with $\lambda = 2ms/R$.

• We can assume $u \leq 1$ on ∂E. Since $\mathbf{\Sigma}$ stoch. compl & Feller, $u \leq h \to 0$ where h is the minimal, positive solution of (▲) on E.

• This means that $f(x) \to 0 \in \mathbb{R}^m$ as $x \to \infty \in E$.

• Move the origin of the extrinsic ball from 0 to 0_ε so that $f(E) \subset B_{R+\varepsilon}(0_\varepsilon)$. Again, $1 - (R+\varepsilon)\sup_E |\mathbf{H}| > 0$ and the previous argument applies. Therefore, $f(x) \to 0_\varepsilon \in \mathbb{R}^m$ as $x \to \infty \in E$. Contradiction. ■