Ricci Curvature and Fundamental Group of Complete Manifolds

(preliminary version)

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Preface

The aim of these lecture notes is to present, in a self-contained form, some classical quantitative estimates of the fundamental group of a complete Riemannian manifold with non-negative Ricci curvature: we start from the fundamental theorem by Bonnet-Myers, we discuss the Švarc-Milnor growth theory, the Cheeger-Gromoll splitting theorem and, finally, we present a more recent estimate by M. Anderson and P. Li concerning non-compact manifolds with non-negative Ricci curvature and maximal volume growth. Some concluding remarks on the role of the Ricci curvature assumption and some variations on the theme of Anderson-Li theorem are also presented. A final Appendix contains basic tools of Geometric Analysis, namely, the Bochner formula, an integral estimate of the Ricci tensor along minimizing geodesics and the Laplacian and the volume comparison theorems.

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Introduction

The are several curvatures on a Riemannian manifold of dimension $m\,(\geq 3)$. The sectional curvature along the 2-plane $X \wedge Y \subset T_{x_0}M$

$$Sec\,(X \wedge Y) = \frac{Riem\,(X,Y,Y,X)}{|X|^2\,|Y|^2 - \langle X,Y\rangle^2},$$

the Ricci curvature in direction $X \in T_{x_0}M$,

$$Ric\,(X,X) = \sum_{j=1}^{m} \; Riem\,(X,E_j,E_j,X)$$

and the scalar curvature

$$Scal = \sum_{j=1}^{m} Ric\,(E_j,E_j)\,,$$

where $\{E_j\}$ denotes any local o.n. frame. Obviously, they are in a precise hierarchy: the sectional curvature contains more information on the underlying Riemannian structure than the Ricci curvature. The scalar curvature is the most elusive of the Riemannian invariants.

There is no local obstruction for a smooth manifold to support a metric whose sectional(!!!) curvatures have a prescribed sign. In fact the following astonishing result holds. A readable proof is contained in the lecture notes [13]. This is actually a result in differential topology. The proof is a beautiful application of the Gromov **h-principle** for open, invariant, partial differential relations.

Theorem 0.1 (Gromov) *Let M be a non-compact, smooth manifold. Then M has a Riemannian metric with $Sec < 0$ and a Riemannian metric with $Sec > 0$.*

Obstructions appear as soon as we impose some extra global property on M, e.g., **compactness** or, more generally, we require some extra global property on the resulting Riemannian space $(M, \langle , \rangle)$, e.g., **geodesic completeness** (or other kind of completeness)**.** All such obstructions have topological nature. Although we will be interested in Ricci curvature, for comparison with Gromov result let us mention a few striking theorems involving the sectional curvature. First, let us consider the negative side.

Theorem 0.2 (Cartan-Hadamard) *Let $(M, \langle , \rangle)$ be a complete manifold of dimension m and with sectional curvature $Sec \leq 0$. Then, having fixed an origin $o \in M$, the exponential map $\exp_o : T_oM \approx \mathbb{R}^m \to M$ has no critical points and therefore it is a (universal) covering map. In particular, M is a so called $K(\pi, 1)$-space, i.e., its homotopy groups satisfy $\pi_j(M) = 0$, for every $j \geq 2$.*

In particular, it follows that a complete simply connected Riemannian manifold of non-positive sectional curvature is diffeomorphic to the Euclidean space. Such a manifold is usually called a **Cartan-Hadamard manifold**.

From a certain viewpoint, the next result is an example of **geometric rigidity of discrete groups**: roughly speaking, let Γ be the fundamental group $\pi_1(M)$ of a Riemannian manifold $(M, \langle , \rangle)$. Say that Γ is rigid if an injective homomorphism (representation) $\rho : \Gamma \to \mathcal{I}(N)$ into the isometry group of a compact/complete manifold of nonpositive curvature is induced by a totally geodesic immersion $f : M \to N$.

Theorem 0.3 (Preissman, Lawson-Yau, Gromoll-Wolf) *Let $(M, \langle , \rangle)$ be a compact manifold with sectional curvature $Sec \leq 0$. If $\pi_1(M)$ contains an isomorphic image of $\mathbb{Z}^2$, then M contains a totally geodesic flat torus $\mathbb{T}^2$. In particular, if $Sec < 0$ then $\pi_1(M)$ does not contain $\mathbb{Z}^2$ as a subgroup.*

Now, let us mention what happens on the positive side. Recall that a subset $C \subset M$ is said to be **totally convex** if, for every $p \neq q \in C$, any geodesic segment of M connecting p with q is contained in C.

Theorem 0.4 (Gromoll-Meyer, Cheeger-Gromoll, Perelman) *Let $(M, \langle , \rangle)$ be a complete, non-compact Riemannian manifold satisfying $Sec \geq 0$. Then, there exists a totally convex, totally geodesic submanifold $S \subset M$ (the **soul** of M) such that M is diffeomorphic to the normal bundle of S in M. Furthermore, if $Sec > 0$ at a point $p \in M$ then $S = \{\text{point}\}$ and M is diffeomorphic to $\mathbb{R}^m$. In particular, if M is compact with non-negative* *sectional curvature and it is a $K(\pi,1)$-space then M is flat and $\pi_1(M)$ is a subgroup of the rigid motions of $\mathbb{R}^m$ (a crystallographic group).*

The following "differentiable sphere theorem" has been proved in 2008 using Ricci-flow methods à la Hamilton, [6]. It answers in the affirmative a conjecture by Klingenberg. The topological version is older and proved using comparison techniques.

Theorem 0.5 (Berger, Klingenberg, Brendle-Schoen) *Let $(M, \langle,\rangle)$ be a compact, simply connected Riemannian manifold of dimension $m \geq 4$ satisfying the $1/4$-pinching curvature condition, namely, $\delta < Sec \leq 4\delta$, for some $\delta > 0$. Then, M is diffeomorphic to the standard(!) sphere $\mathbb{S}^m$.*

In the Ricci curvature perspective we have the following theorem that encloses some (recent and) very surprising results from [30], [31].

Theorem 0.6 (Lohkamp) *Let M be a smooth (possibly compact) manifold of dimension $\dim M = m \geq 3$. Then M can be equipped with a complete metric $\langle,\rangle$ with Ricci curvature satisfying $-a(m) \leq Ric \leq -b(m) < 0$. Furthermore, if M is non-compact, then every Riemannian metric $\langle,\rangle$ on M can be pointwise conformally deformed to a complete metric $\widetilde{\langle,\rangle} = \lambda^2(x)\langle,\rangle$, $\lambda > 0$, such that $\widetilde{Ric} < 0$.*

In sharp contrast, by reversing the Ricci curvature bound, i.e. $Ric \geq 0$, a number of topological obstructions immediately arise: these involve both the fundamental group and the homology groups of the underlying manifold. The obstructions related to the fundamental group represent the subject of the present lectures and will be analyzed in details in the next chapters. A special attention will be devoted to quantitative aspects of the theory, such as estimates of the cardinality and, more generally, of the asymptotic growth of the fundamental group. In each of the considered situations we shall also illustrate in what sense the quantitative bounds can be considered sharp.

For the sake of completeness, let us conclude this introductory chapter by mentioning very briefly what happens for the weakest of the Riemannian invariants, namely, the scalar curvature. Clearly, as a consequence of Lohkamp result, there is no obstruction for a manifold of dimension $m \geq 3$ (either open or closed) to support a complete metric with $Scal < 0$. In the opposite direction, once again, the topology of the underlying manifold plays a key role. Both the obstruction theory and the existence theory for complete metrics with $Scal \geq 0$ (> 0) is nowadays the subject of intensive (and very difficult) research. In light of the fact that the scalar curvature contains subtle information, the topology must be detected by more sophisticated invariants. Ground-braking results were obtained by R. Schoen and S.-T. Yau, [43], [44], and by M. Gromov and B. Lawson, [17], [18], [19]. We limit ourselves to quote the following theorem concerning aspherical manifolds, i.e., manifolds with contractible universal covering.

Theorem 0.7 (Schoen-Yau, Gromov-Lawson) *Let M be an aspherical manifold (possibly compact) which admits a complete Riemannian metric with finite volume and $-a \leq Sec \leq -b < 0$. Then, M does not support any complete Riemannian metric with $Scal > 0$.*

Some notation

In the sequel, $(M, \langle,\rangle)$ will always denote a complete Riemannian manifold of dimension $\dim M = m$. The corresponding intrinsic distance will be denoted by d or d_M. As usual, this latter gives rise to a “distance” between subsets $A, B \subset M$ defined by $\mathrm{dist}(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$.

Having fixed a reference origin $o \in M$, we set $B_r(o)$ for the open metric ball of M centered at $o \in M$ and of radius $r > 0$. The corresponding geodesic sphere is $\partial B_r(o)$. The Riemannian measure of $(M, \langle,\rangle)$ is represented by $d\mathrm{Vol}$ so that $\mathrm{Vol}(B_r(o)) = \int_{B_r(o)} d\mathrm{Vol}$. Furthermore, $d\mathrm{Vol}$ gives rise to a $(m-1)$-dimensional Hausdorff measure, say $d\mathrm{Area}$, and the area of the geodesic sphere is given by $\mathrm{Area}(\partial B_r(o)) = \int_{\partial B_r(o)} d\mathrm{Area}$. By the co-area formula $\mathrm{Vol}(B_R(o)) = \int_0^R \mathrm{Area}(\partial B_t(o))\, dt$.

As above, let $o \in M$ be a fixed origin. The cut locus of o is denoted by $\mathrm{cut}(o)$. Thus, the distance function $r_o(x) = d(x, o)$ is smooth on the open set $M \backslash \mathrm{cut}(o)$ of full measure in M, and $|\nabla r_o| = 1$. We shall often use the symbol $r(x)$ instead of $r_o(x)$.

Chapter 1

Universal covering and fundamental group: the Bonnet-Myers theorem

The fundamental group of a space, the notions of group action and orbit space, and the concept of (topological) covering are so tightly related that can be considered as different faces of the same object.

Suppose we are given a **discrete group** Γ (countable group with the discrete topology) acting smoothly on smooth manifold N. The quotient topological space N/Γ with respect to the obvious equivalence relation

$$x \sim y \Longleftrightarrow y = \gamma x, \text{ for some } \gamma \in \Gamma,$$

is called the **orbit space**. Recall that the action is:

(a) **free** (of fixed points) if $\gamma x = x$, for some x implies $\gamma = 1$.

(b) **proper** if, for every compact set $K \subset N$, $\gamma K \cap K \neq \emptyset$ for at most finitely many $\gamma \in \Gamma$.

Since manifolds are Hausdorff spaces, it can be shown that a free+proper action is

(c) **properly discontinuous**, i.e., for any fixed $x \in N$, there exists a neighborhood U such that $\gamma_1 U \cap \gamma_2 U = \emptyset$, for every $\gamma_1 \neq \gamma_2 \in \Gamma$.

Discrete group actions appear naturally in the setting of covering maps. Let $P : N \to M$ be a smooth covering, and let Γ denote its topological covering transformation group. By definition Γ is the group made up by all the fiber-preserving homeomorphisms of N, i.e.,

$$P \circ \gamma = P, \ \forall \gamma \in \Gamma.$$

From this latter we see that, actually, $\Gamma \subset \mathrm{Diff}(N)$. In fact, suppose also that $(N, \langle , \rangle_N)$ and $(M, \langle , \rangle_M)$ are Riemannian manifolds and P is a **Riemannian covering map**, that is $\langle , \rangle_N = P^* \langle , \rangle_M$. Then the above relation implies $\Gamma \subset \mathcal{I}(N)$, the isometry group of $(N, \langle , \rangle_N)$.

The covering transformation group Γ acts naturally on N by evaluation:

$$\gamma x = \gamma(x).$$

We say that such an action is

(d) **transitive on the fibre** if, for every $x_1, x_2 \in P^{-1}(x)$, there exists $\gamma \in \Gamma$ such that $\gamma(x_1) = x_2$.

In the following quite comprehensive result we gather (e.g. from [26], [27], [36]) some of the basic relations between coverings, fundamental group, group actions and Riemannian structure. More group action theory will be recalled in a subsequent section.

Theorem 1.1 *(A) Let $(N, (,))$ be a Riemannian manifold and let $\Gamma \subset \mathcal{I}(N)$ be a discrete group acting freely and properly on N. Then, N/Γ with the quotient topology has a unique differentiable structure such that the quotient map $P : N \to N/\Gamma$ is a smooth covering map. Furthermore the Γ-invariant metric $(,)$ descends to a Riemannian metric $(,)_{N/\Gamma}$ on N/Γ so that P is a local isometry (hence a Riemannian covering). In particular*

$$\mathrm{Curv}_N(x) = \mathrm{Curv}_{N/\Gamma}(P(x))$$

and N/Γ is geodesically complete if and only if N is geodesically complete. The fundamental group $\pi_1(N/\Gamma)$ of the orbit space N/Γ satisfies

$$\pi_1(N/\Gamma)/H_0 \simeq \Gamma \simeq \mathrm{Deck}\left(N \stackrel{P}{\to} N/\Gamma\right),$$

where

$$H_0 = P_{\#}(\pi_1(N)) \lhd \pi_1(N/\Gamma),$$

and $P_{\#_x} : \pi_1(N, x) \to \pi_1(N/\Gamma, \Gamma x)$ *is the induced homomorphism. Furthermore, suppose that* N *(hence* N/Γ*) is complete. Then, the intrinsic metric* $d_{N/\Gamma}$ *is the subset-distance with respect to* d_N*, i.e.,*

$$d_{N/\Gamma}(\Gamma x, \Gamma y) = \mathrm{dist}_N(\Gamma x, \Gamma y) = \mathrm{dist}_N(\{x\}, \Gamma y) = \mathrm{dist}_N(\Gamma x, \{y\}),$$

and, for every $x_0 \in N$ *and* $r > 0$, *it holds*

$$P\left({}^N B_r(x_0)\right) = {}^{N/\Gamma} B_r(P(x_0)).$$

(B) Conversely, let $P : (N, \langle,\rangle_N) \to (M, \langle,\rangle_M)$ *be a Riemannian covering map (with possibly* N *non-simply connected). Assume that it is a* ***normal covering****, i.e.,*

$$P_{\#}\pi_1(N) \lhd \pi_1(M).$$

Let Γ *be the group of topological, hence isometric, covering transformations. Then* $\Gamma \subset \mathcal{I}(N)$ *is a discrete group acting freely and properly on* N *and part (A) of the statement applies. Furthermore, since the covering is normal, the action is transitive on the fibres and it follows that* N/Γ *is isometric to* M *via the map induced by* P *on the quotient manifold* N/Γ*.*

(C) Let $P : (N, \langle,\rangle_N) \to (M, \langle,\rangle_M)$ *be a local isometry between Riemannian manifolds. If* $(N, \langle,\rangle_N)$ *is complete, then* $(M, \langle,\rangle_M)$ *is complete,* P *is a Riemannian covering map and part (B) of the statement applies.*

Let us specialize the previous theorem to the simply connected case. Let $(M, \langle,\rangle)$ be a Riemannian manifold of dimension $\dim M = m$. The universal covering of M will be always denoted by $P : M' \to M$. The simply connected covering manifold M' is endowed with the pull back metric $\langle,\rangle' = P^*\langle,\rangle$ so that P becomes a Riemannian covering map. Note that, by Cartan-Hadamard theorem, if M and hence M' are geodesically complete and ${}^M Sec \leq 0$, we can always think of $M' = T_o M$ and

$$P = \exp_o : T_o M \to M$$

the exponential map from a fixed origin $o \in M$. Let $\Gamma \subset \mathcal{I}(M')$ be the covering transformation group. Then, the quotient manifold M'/Γ (with the induced metric) is isometric to M. Accordingly, we shall always identify

$$M = M'/\Gamma$$

so that

$$P^{-1}(x_0) = \Gamma x_0,$$

the Γ-orbit of $x_0 \in M$. In particular, it follows that

$$\left|P^{-1}(x_0)\right| = |\Gamma|, \forall x_0 \in M.$$

Finally, since M' is simply connected, we have a canonical isomorphism

$$\Gamma \simeq \pi_1\left(M'/\Gamma, x_0\right) = \pi_1\left(M, x_0\right).$$

In what follows we shall always identify these two groups

$$\Gamma = \pi_1\left(M'/\Gamma, x_0\right).$$

To begin with, we prove the following classical result.

Theorem 1.2 (Bonnet-Myers) *Let $(M, \langle,\rangle)$ be a complete, m-dimensional manifold with $Ric \geq (m-1)\, c^2 > 0$. Then*

(a) M is compact with $\operatorname{diam}(M) \leq \pi/c$*;*

(b) $|\pi_1(M)| < +\infty$.

Proof. To prove (a), suppose by contradiction that there are $o, x \in M$ such that $d(o,x) > \pi/c$. let $\gamma : [0, L] \to M$ be a unit speed, minimizing geodesic from $\gamma(0) = o$ to $\gamma(L) = x$. Since γ is minimizing, the second variation formula for arch-length implies that, for every Lipschitz function $h : [0, L] \to \mathbb{R}$ satisfying $h(0) = h(L) = 0$ it holds

$$0 \leq \int_0^L \left\{ \left(h'(s)\right)^2 - \frac{Ric\left(\dot\gamma(s), \dot\gamma(s)\right)}{m-1} h^2(s) \right\} ds.$$

Actually, this latter inequality can be derived directly from the Bochner formula for the distance function, thus avoiding the use of arc-length variations; see Lemma A.2 in the Appendix. Using the Ricci bound we obtain

$$\int_0^L \left\{ \left(h'(s)\right)^2 - \frac{Ric\left(\dot\gamma(s), \dot\gamma(s)\right)}{m-1} h^2(s) \right\} ds \leq \int_0^L \left\{ \left(h'(s)\right)^2 - c^2 h^2(s) \right\} ds.$$

Consider

$$h(t) = \sin\left(\frac{\pi t}{L}\right).$$

Since $L > \pi/c$, direct computations show that

$$\int_0^L \left\{ \left(h'(s)\right)^2 - c^2 h^2(s) \right\} ds < 0,$$

contradiction.

To prove (b), we introduce the Riemannian universal covering map $P: M' \to M$, which is a local isometry, hence preserves curvature and completeness. Since $(M', \langle , \rangle')$ is a complete manifold with the same Ricci lower bound of M it follows that M' is compact, hence, $P^{-1}(x_0)$ is finite, proving that $|\pi_1(M, x_0)| < +\infty$. ■

Example 1.3 *Consider the* m*-dimensional torus realized as the orbit space* $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$*, where* $\mathbb{Z}^m$ *acts by translations along the axis. Then* $\pi_1(\mathbb{T}^m) = \mathbb{Z}^m$ *proving that* $\mathbb{T}^m$ *has no Riemannian metrics with Ric* > 0*. Note that, obviously,* $\mathbb{T}^m$ *has the flat (hence Ricci-flat) metric induced by* $\mathbb{R}^m$ *on the orbit space.*

Example 1.4 *Let* N *be any compact manifold of dimension* $\dim N = m \geq 3$*. Consider the connected sum* $M = \mathbb{T}^m \# N$*. Using Seifert-Van Kampen theorem, it is easy to show that* $\pi_1(M)$ *contains an isomorphic image of* $\mathbb{Z}^m$*. More precisely,*

$$\pi_1(M) = \pi_1(\mathbb{T}^m) * \pi_1(N)$$

where $*$ *denotes the free product of groups; see Chapter 2. Therefore,* $|\pi_1(M)| = +\infty$ *and it follows form Bonnet-Myers that* M *has no metrics with Ric* > 0*. Actually, in case e.g.* $\pi_1(N)$ *contains an element of infinite order, we shall see that such a manifold* M *does not support any metric with Ric* ≥ 0*.*

Problem 1.5 *In the assumptions of Bonnet-Myers, is it possible to estimate* $|\pi_1(M)|$ *in terms of the Ricci curvature lower bound?*

Example 1.6 *Consider the standard sphere*

$$\mathbb{S}^3 = \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1 \right\} \subset \mathbb{C}^2,$$

endowed with its standard metric of constant curvature $+1$*. For every* $k \in \mathbb{N}$ *fix* $h \in \mathbb{N}$ *with* $\mathrm{MCD}(k, h) = 1$*. The additive group* $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$ *acts by isometries on* $\mathbb{S}^3$ *as follows*

$$[m]_k(z_1, z_2) = \left(e^{i\frac{2\pi}{k}m} z_1, e^{i\frac{2\pi h}{k}m} z_2 \right).$$

The action is free and obviously proper (the group is finite). Therefore, the quotient space

$$\mathbb{L}(k, h) = \mathbb{S}^3/\mathbb{Z}_k$$

is a smooth, compact 3-dimensional manifold called a ***lens space****. The projection* $P : \mathbb{S}^3 \to \mathbb{L}(k,h)$ *is a* k*-fold universal covering map and the standard metric of* $\mathbb{S}^3$ *descends to a Riemannian metric on* $\mathbb{L}(k,h)$*. Accordingly,* $\mathbb{L}(k,h)$ *has constant (sectional) curvature* $+1$*. On the other hand* $\pi_1(\mathbb{L}(k,h)) = \mathbb{Z}_k$*, and it follows that* $|\pi_1(\mathbb{L}(k,h))| \to +\infty$ *as* $k \to +\infty$*. This answers in the negative the previous question.*

The following elementary fact can be considered as a quantitative version of the finiteness part of Bonnet-Myers theorem. Later on, following Anderson paper [3], we shall extend this conclusion to complete(!) manifolds with non-negative Ricci curvature and maximal volume growth. In what follows $\mathbb{S}^m_c$ will denote the sphere of constant sectional curvature $c > 0$ endowed with its standard metric. Thus, $\mathbb{S}^m = \mathbb{S}^m_1$.

Theorem 1.7 *Let* $(M, \langle , \rangle)$ *be a complete (hence compact),* m*-dimensional Riemannian manifold satisfying* $Ric \geq (m-1)c > 0$*. Then*

$$|\pi_1(M)| \leq \frac{\mathrm{Vol}(\mathbb{S}^m_c)}{\mathrm{Vol}(M)},$$

and the equality holds if and only if $M = \mathbb{S}^m_c / \pi_1(M)$[1]*. In particular, if* $\mathrm{Vol}(\mathbb{S}^m_c) < 2\mathrm{Vol}(M)$ *then* M *is simply connected.*

In order to prove this result, we shall employ two important tools in geometric analysis: the area formula and the (ubiquitous) Bishop-Gromov comparison theorem.

Recall that, given a smooth function $\varphi : M \to N$ between manifolds of the same dimension, the set of critical values $\mathcal{L}_\varphi$ of φ is a set of measure zero in N (Sard theorem). In particular, by the inverse function theorem, for every $y \in N \backslash \mathcal{L}_\varphi$, the fiber $\varphi^{-1}(y)$ is a discrete, (at most) countable set[2].

---

[1] Compact manifolds of positive constant curvature $M = \mathbb{S}^m_c/\Gamma$, where $\Gamma \subset \mathcal{I}(\mathbb{S}^m_c)$ is a discrete group, have been classified; see e.g. [49].

[2] If M is compact and N is connected, we have: (a) $|\varphi^{-1}(y)| < +\infty$ for every $y \notin \mathcal{L}_\varphi$, (b) the function $y \longmapsto |\varphi^{-1}(y)|$ is locally constant and (c) the $\mathbb{Z}_2$-class of $|\varphi^{-1}(y)|$ does not depend on the choice of the regular value y. One defines $\deg_2(\varphi) = |\varphi^{-1}(y)| \pmod 2$ called the **mod 2-degree.** If both M and N are oriented and φ preserves the orientation then $|\varphi^{-1}(y)|$ itself does not depend on $y \notin \mathcal{L}_\varphi$. More generally, for every regular point x, let $\mathrm{sgn}(\varphi)(x) = \pm 1$ according to the fact that φ preserves or reverses the orientation. Then $\deg(\varphi) = \sum_{x \in \varphi^{-1}(y)} \mathrm{sgn}(\varphi)(x)$ does not depend on $y \notin \mathcal{L}_\varphi$ and it is called the **Brouwer degree**. The degree (both mod 2 and Brouwer) depends only on the homotopy class of φ; [32].

Theorem 1.8 (Area formula) *Suppose we are given a smooth (Lipschitz is enough) map* $\varphi : M \to N$ *between Riemannian manifolds of the same dimension* $\dim M = \dim N = m$ *and let* $f \in C^0(M) \cap L^1(M)$. *Define*

$$N_f(\varphi)(y) = \sum_{x \in \varphi^{-1}(y)} f(x) : N \backslash \mathcal{L}_\varphi \to \mathbb{R}.$$

Then

$$\int_N N_f(\varphi)(y)\, d\mathrm{Vol}(y) = \int_M f(x)\, |d_x\varphi|\, d\mathrm{Vol}(x).$$

In Chapter 4 below we shall provide a direct proof of the Area formula in the special case $f(x) = 1$ and φ is a Riemannian covering map between compact manifolds. That proof, in a sense, will suggest how Theorem 1.7 can be extended to non-compact situations. We now state the volume comparison by Bishop-Gromov. For a proof, see the Appendix.

Theorem 1.9 (Bishop-Gromov) *Let* $(M, \langle,\rangle)$ *be a complete Riemannian manifold of dimension* $\dim M = m$ *satisfying*

$$Ric \geq (m-1)\, c, \text{ on } M$$

for some constant $c \in \mathbb{R}$. *Denote by* $\mathbf{B}_r(0)$ *the geodesic ball of radius* $r > 0$ *and centered at the origin of the* m*-dimensional space-form* $\mathbf{M}^m(c)$ *of constant sectional curvature* c. *Then, the functions*

$$r \longmapsto \frac{\mathrm{Area}(\partial B_r(o))}{\mathrm{Area}(\partial \mathbf{B}_r(0))} \quad \text{and} \quad r \longmapsto \frac{\mathrm{Vol}(B_r(o))}{\mathrm{Vol}(\mathbf{B}_r(0))}$$

are non-increasing. In particular

$$\mathrm{Vol}(B_r(o)) \leq \mathrm{Vol}(\mathbf{B}_r(0)),\ \forall r > 0$$

and the equality holds for some $R_0 > 0$ *if and only if* $B_{R_0}(o)$ *is isometric to* $\mathbf{B}_{R_0}(0)$.

We are now in the position to give the
Proof (of Theorem 1.7). Without loss of generality, take $c = 1$. We introduce the Riemannian universal covering $P : M' \to M = M'/\Gamma$ where $\Gamma = \pi_1(M)$. By Bonnet-Myers, M' is compact, $\mathrm{diam}(M') \leq \pi$ and $|\Gamma| < +\infty$. Using the Area formula with $\varphi = P$ and $f(x) = 1$ we deduce

$$|\Gamma|\, \mathrm{Vol}(M) = \mathrm{Vol}(M').$$

On the other hand, since $Ric \geq (m-1)$, the Bishop-Gromov comparison applied to a geodesic ball of radius $r = \pi$ implies

$$\mathrm{Vol}\left(M'\right) \leq \mathrm{Vol}\left(\mathbb{S}^m\right),$$

and therefore

$$|\Gamma| \,\mathrm{Vol}\left(M\right) = \mathrm{Vol}\left(M'\right) \leq \mathrm{Vol}\left(\mathbb{S}^m\right),$$

as desired. In case $|\Gamma| = \mathrm{Vol}\left(\mathbb{S}^m\right)/\mathrm{Vol}\left(M\right)$, from the above we get $\mathrm{Vol}\left(M'\right) = \mathrm{Vol}\left(\mathbb{S}^m\right)$ and from the equality case in the Bishop-Gromov comparison theorem we deduce that M' is isometric to $\mathbb{S}^m$. ■

In case of manifolds (even compact) for which the curvature assumption is relaxed to $Ric \geq 0$, we immediately realize that we have to switch from finite to infinity fundamental groups and the purpose now becomes to get some estimates on the "growth" of this group in a sense to be made precise. This requires a small number of notions from geometric group theory ("groups as geometric objects") and a more careful analysis of discrete group actions. Such a viewpoint was introduced by the Russian mathematicians Efremovich and Švarc and, later, by Milnor.

Chapter 2

Finitely generated groups as geometric objects

For this section, we essentially follow the presentations in [21] and [33]. Beside the seminal books [15] and [16], further (advanced) aspects, oriented especially towards hyperbolicity and non-positive curvature, are dealt in [5].

One has two players: a group (typically finitely generated and discrete) and a metric space (e.g. a Riemannian manifold or, more generally, a "geodesic" space). The group acts by isometries on the space. Actually, the group itself is a metric space with respect to a distance constructed via a set of generators. Geometric Group Theory relates the algebraic properties of the acting group to the geometry of the orbit-space. The geometry of the orbit space, in turn, is often reflected into the geometry of the group seen itself as a metric space.

2.1 Generators and presentations

Definition 2.1 *A* ***set of generators*** *for a group Γ is a subset $S \subseteq \Gamma$ with the following property: for every $\gamma \in \Gamma$ there exists a finite sequence $s_1, ..., s_k \in S \cup S^{-1}$ such that $\gamma = s_1 \cdots s_k$. In this case, we write $\Gamma = \langle S \rangle$.*

Clearly every group has a set of generators, possibly $S = \Gamma$. In general neither the representation $\gamma \in \Gamma$ in terms of the generators is unique nor S is finite. The uniqueness problem leads to the concept of free group.

Definition 2.2 *Let $\Gamma = \langle S \rangle$. Any finite product $s_1 \cdots s_k$ with $s_j \in S \cup S^{-1}$ is called a* ***word****. Such a word is said to be* ***reduced*** *if it does not contains* *neither* 1 *nor any couple of the form* $s_j s_j^{-1}$. *In case each element* $\gamma \in \Gamma$ *writes uniquely as a reduced word in the symbols of* $S \cup S^{-1}$ *then we say that* Γ *is a* ***free group*** *with set of* ***free generators*** S *(equivalently,* Γ *is freely generated by* S*). The* ***rank*** *of the free group* $\Gamma = \langle S \rangle$ *is the cardinality* $|S| \leq +\infty$.

Example 2.3 *The additive group* $\mathbb{Z}$ *is freely generated by* $S = \{1\}$ *and* $S = \{-1\}$. *However, for any choice of* $q, p \in \mathbb{Z}$ *with* $MCD\,(q,p) = 1$, *the additive group* $\mathbb{Z}$ *is generated, but not freely generated, by* $S = \{q, p\}$. *The first claim follows from the fact the Diophantine equation* $xq + yp = 1$ *has a solution* $(x_0, y_0) \in \mathbb{Z} \times \mathbb{Z}$. *But, in fact, such equation has infinitely many solutions and this proves also the second claim.*

The importance of free groups derives from the next

Theorem 2.4 *Let* $\Gamma = \langle S \rangle$ *be any group. Then, there exists a free group* $F(S)$ *on the set of symbols(!)* S *and a surjective homomorphism* $\overline{j} : F(S) \to \Gamma$. *In particular,* $\Gamma \simeq F(S) / \ker\left(\overline{j}\right)$.

Proof (outline). Let $W(S)$ be the set of all the formal[1] words in the symbols $S \cup S^{-1}$. Introduce in $W(S)$ the equivalence relation which identifies two words $w_1 \sim w_2$ provided we can pass from w_1 to w_2 by adding or deleting a finite sequence of couples ss^{-1} or $s^{-1}s$ with $s \in S$. A word that does not contain such a couple is said to be reduced. Let $F(S) = W(S) / \sim$. It can be verified that $F(S)$ is a group with respect to the juxtaposition (the proof of associativity is tricky). The unity is the (class of the) empty word and the inverse of (the class of) $w = s_1 \cdots s_k$ is (the class of) $w^{-1} = s_k^{-1} \cdots s_1^{-1}$. By construction, $F(S)$ is freely generated by S. Using the uniqueness representation of an element of $F(S)$ as a reduced word, it is now easy to verify that the following extension property holds:

let $h : S \to H$ *be a map into a group* H. *Then* h *extends to a unique homomorphism* $\overline{h} : F(S) \to H$.

To see this, for every $w \in F(S)$, express uniquely $w = s_1 \cdots s_k$ and take $\overline{h}(w) = h(s_1) \cdots h(s_k)$.

Thus, in particular, the inclusion $j : S \hookrightarrow \Gamma = \langle S \rangle$ extends to a unique surjective homomorphism $\overline{j} : F(S) \to \Gamma$. By the first isomorphism theorem, $\Gamma \simeq F(S) / \ker\left(\overline{j}\right)$. ■

[1] without the meaning they would have in Γ. For instance, even in case $a, b, c \in S$ and $ab = c$ in Γ, we do not have $ab = c$ in $W(S)$.

Definition 2.5 *In the above situation, let R be a set of generators of* $\ker\left(\bar{j}\right)$. *Then, $\langle S|R\rangle$ is called a* ***presentation*** *of* Γ.

The above theorem states precisely that every group has a presentation. By means of presentations it is easy to give an "operative" definition of free product. To this aim, note that a kind of "converse" of Theorem 2.4 holds, namely, given a set of symbols S and a set R of (reduced) words in the symbols $S \cup S^{-1}$ there exists a group Γ presented by $\langle S|R\rangle$. To obtain Γ, first construct the abstract free group $F(S)$ and, then, take $N(R) \lhd F(S)$ as the smaller, normal subgroup of $F(S)$ generated by R. Clearly, $N(R)$ is the intersection of all the normal subgroups of $F(S)$ containing R. Thus, $\Gamma = F(S)/N(S)$.

Definition 2.6 *Let $\Gamma_1 = \langle S_1|R_1\rangle$ and $\Gamma_2 = \langle S_2|R_2\rangle$ be given presented group. The* ***free product*** *of Γ_1 and Γ_2 is defined as the group presented by $\Gamma_1 * \Gamma_2 = \langle S_1 \cup S_2|R_1 \cup R_2\rangle$ where no further relation between generators in S_1 and S_2 occure*[2].

We now consider the second question.

Definition 2.7 *A group Γ is said to be* ***finitely generated*** *if it has a finite set of generators S.*

The finitely generated group $\Gamma = \langle S\rangle$ is said to be ***finitely presented*** *if also* $\ker\left(\bar{j}\right)$ *is finitely generated. In this case, we have a finite presentation $\langle S|R\rangle$ with R a finite set of generators of* $\ker\left(\bar{j}\right)$.

Example 2.8 *The additive group $\mathbb{Z}$ is finitely generated whereas the additive group $\mathbb{Q}$ is not. Needles to say, even if Γ is finitely generated, one can specify an infinite set of generators. For instance $\mathbb{Z} = \langle S\rangle$ where $S = \{p \in \mathbb{N} : p \ \text{ prime}\}$. Note however that we can extract from S a finite set of generators (any couple of distinct primes). This is in fact a general result.*

Let $\Gamma = \langle S\rangle$ be a finitely generated group. Then, there exists $S' \subset S$ such that $|S'| < +\infty$ and $\Gamma = \langle S'\rangle$.

To see this, simply fix a finite set of generators $S'' \subset \Gamma$ and note that each $s'' \in S''$ expresses as a product of (finitely many) s, s^{-1} with $s \in S$. The collection of these elements "s" is the desired finite set S'.

[2] Note that, with this formalism, the direct product $\Gamma_1 \times \Gamma_2$ of $\Gamma_1 = \langle S_1|R_1\rangle$ and $\Gamma_2 = \langle S_2|R_2\rangle$ is the group presented as $\Gamma_1 \times \Gamma_2 = \langle S_1 \cup S_2|R_1 \cup R_2 \cup R\rangle$, where $R = \{s_1 s_2 = s_2 s_1 : s_1 \in S_1, s_2 \in S_2\}$.

Remark 2.9 *In the definition of finitely presented groups, we require(!) that the subgroup* $\ker(\bar{j})$ *of the finitely generated free group* $F(S)$ *is itself finitely generated. This is not automatic in view of the following classical result.*

Theorem 2.10 (Higman-Neumann-Neumann) *Every countable group has an isomorphic image into a group with two generators.*

2.2 The word metric

We now turn groups into metric spaces.

Definition 2.11 *Suppose we are given* $\Gamma = \langle S \rangle$. *For every* $\gamma \in \Gamma$, $\gamma \neq 1$, *define the **word length of*** γ *(with respect to* S*) as*

$$\ell_S(\gamma) = \min\left\{k \in \mathbb{N} : \gamma = s_1 \cdots s_k,\ s_j \in S \cup S^{-1}\right\}.$$

Set also $\ell_S(1) = 0$. *This gives rise to a function* $d_S : \Gamma \times \Gamma \to \mathbb{N}$ *given by*

$$d_S(\gamma_1, \gamma_2) = \ell_S\left(\gamma_1^{-1}\gamma_2\right)$$

*which is called the **word metric** of* Γ *with respect to the set of generators* S.

As an immediate consequence of the above definition, we have

Lemma 2.12 *The word length function satisfies the following properties:*

(a) $\ell_S(\gamma) = 0$ *if and only if* $\gamma = 1$.

(b) ℓ_S *is subadditive, i.e.,* $\ell_S(\gamma\gamma') \leq \ell_S(\gamma) + \ell_S(\gamma')$.

(c) $\ell_S(\gamma) = \ell_S\left(\gamma^{-1}\right)$.

In particular, the word metric d_S *is a genuine distance and* $\Gamma = \langle S \rangle$ *is a subgroup of the isometry group of* (Γ, d_S).

Proof. Property (a) holds by definition. On the other hand, if $\ell_S(\gamma) = k$ and $\ell_S(\gamma') = k'$, so that $\gamma = s_1 \cdots s_k$ and $\gamma' = s_{1'} \cdots s_{k'}$, then $\ell_S(\gamma\gamma') \leq k + k' = \ell_S(\gamma) + \ell_S(\gamma')$. This shows the validity of (b). Finally, consider $\gamma = s_1 \cdots s_k$, with $k = \ell_S(\gamma)$. Then $\gamma^{-1} = s_k^{-1} \cdots s_1^{-1}$, proving that $\ell_S\left(\gamma^{-1}\right) \leq k$. Changing the role of γ and γ^{-1} yields (c).

The second part of the statement now follows by interpreting properties (a)–(c) in terms of d_S. ■

Since d_S takes values in $\mathbb{N}$ it could appear quite surprisingly if one was able to deduce some interesting information on the resulting metric space structure. Despite of this, one can in fact argue very interesting properties of Γ using the word metric. We shall analyze only few of them. First of all, we point out the following

Lemma 2.13 *Let Γ be a finitely generated group and let S, S' be two finite sets of generators of Γ. Then, the identity map $i : (\Gamma, d_S) \to (\Gamma, d_{S'})$ is a bilipschitz homoeomorphism, i.e., there exists $\lambda \geq 1$ such that*

$$\lambda^{-1} d_S(\gamma_1, \gamma_2) \leq d_{S'}(i(\gamma_1), i(\gamma_2)) \leq \lambda d_S(\gamma_1, \gamma_2),$$

for every $\gamma_1, \gamma_2 \in \Gamma$.

Proof. By Lemma 2.12, it suffices to show that there exists $\lambda \geq 1$ satisfying

$$\lambda^{-1} \ell_S(\gamma) \leq \ell_{S'}(\gamma) \leq \lambda \ell_S(\gamma),$$

for every $\gamma \in \Gamma$. To this end, define

$$\begin{aligned} \lambda_1 &= \max\{\ell_{S'}(s) : s \in S\} \\ \lambda_2 &= \max\left\{\ell_S(s') : s' \in S'\right\} \\ \lambda &= \max\{\lambda_1, \lambda_2\} \geq 1. \end{aligned}$$

We claim that λ has the desired property. Indeed, let $\gamma = s_1 \cdots s_k$ with $k = \ell_S(\gamma)$. Then, by subadditivity,

$$\ell_{S'}(\gamma) \leq \sum_{j=1}^{k} \ell_{S'}(s_j) \leq \lambda_1 k = \lambda_1 \ell_S(\gamma) \leq \lambda \ell_S(\gamma).$$

Changing the role of S and S' yields the second inequality. ■

Thus, there is a well defined bilipschitz class on a finitely generated group which does not depend on the chosen finite set of generators. In the sequel, we shall focus our attention on finitely generated groups.

2.3 The fundamental lemma in geometric group theory

Bilipschitz homeomorphisms are very special cases of quasi-isometries. These are perturbations of isometries. Recall that an isometric embedding is a distance preserving (hence injective) map between metric spaces. The isometric embedding is an isometry if it is also surjective. We are going to perturb both these properties.

Definition 2.14 (Gromov) *Let (X, d) and (X', d') be metric spaces.*

(a) *(Perturbation of the distance-preserving property) Let $\lambda \geq 1, L \geq 0$. By a (λ, L)-**quasi-isometric embedding** of X into X' we mean a map $\phi : X \to X'$ such that*

$$\lambda^{-1} d(x, y) - L \leq d'(\phi(x), \phi(y)) \leq \lambda d(x, y) + L,$$

for every $x, y \in X$.

(b) *(Perturbation of surjectivity) A quasi-isometric embedding $\phi : X \to X'$ is said to be a **quasi-isometry** if there exists $D \geq 0$ such that, for every $x' \in X'$ we can find $x \in X$ satisfying $d'(x', \phi(x)) \leq D$. In this case, we say that X is **quasi isometric** to X'.*

It should be noted that "being quasi-isometric" is an equivalence relation. This follows from the next lemma where the concept of quasi-isometry is formulated in terms of a kind of inverse map. The starting point is now that (X, d) and (X', d') are isometric if there are distance preserving maps $\phi : X \to X'$ and $\psi : X' \to X$ such that $\psi \circ \phi = \mathrm{id}_X$ and $\phi \circ \psi = \mathrm{id}_{X'}$.

Lemma 2.15 *The metric spaces (X, d) and (X', d') are quasi-isometric if and only if there exist maps $\phi : X \to X'$ and $\psi : X' \to X$ satisfying the following conditions:*

(a) *(Perturbation of the distance-preserving property) There are constants $\lambda \geq 1, L \geq 0$ such that*

$$d'(\phi(x), \phi(y)) \leq \lambda d(x, y) + L \text{ and } d\left(\psi\left(x'\right), \psi\left(y'\right)\right) \leq \lambda d'\left(x', y'\right) + L,$$

for every $x, y \in X$ and for every $x', y' \in X'$.

(b) *(Perturbation of being an inverse) There is a constant $D \geq 0$ such that*

$$d(\psi \circ \phi(x), \mathrm{id}_X(x)) \leq D \text{ and } d'\left(\phi \circ \psi\left(x'\right), \mathrm{id}_{X'}\left(x'\right)\right) \leq D.$$

Proof. Assume that $\phi : X \to X'$ is a (λ, L)-quasi isometry. For every $x' \in X'$, choose $x = x(x') \in X$ such that $d'(x', \phi(x)) \leq D$. These choices give rise to a map $\psi : X' \to X$ with the desired properties. Indeed, taking $x = x(x')$ and $y = y(y')$ we have

$$\begin{aligned} d\left(\psi\left(x'\right), \psi\left(y'\right)\right) &= d\left(x, y\right) \\ &\leq \lambda d'\left(\phi\left(x\right), \phi\left(y\right)\right) + \lambda L \\ &\leq \lambda\left\{d'\left(\phi\left(x\right), x'\right) + d'\left(x', y'\right) + d'\left(\phi\left(y\right), y'\right)\right\} + \lambda L \\ &\leq \lambda d'\left(x', y'\right) + \lambda\left(2D + L\right), \end{aligned}$$

and

$$d'\left(\phi \circ \psi\left(x'\right), x'\right) = d'\left(\phi\left(x\right), x'\right) \leq D.$$

Moreover, for every $x \in X$, let $\overline{x}' = \phi(x)$ and let $\overline{x} = \psi(\overline{x}') \in X$ so that $d'(\phi(\overline{x}), \overline{x}') \leq D$. Then

$$\begin{aligned} d\left(\psi \circ \phi\left(x\right), x\right) &= d\left(\overline{x}, x\right) \\ &\leq \lambda\left\{d'\left(\phi\left(\overline{x}\right), \phi\left(x\right)\right) + L\right\} \\ &= \lambda\left\{d'\left(\phi\left(\overline{x}\right), \overline{x}'\right) + L\right\} \\ &\leq \lambda\left\{D + L\right\}. \end{aligned}$$

This completes the first part of the proof. The converse implication follows from similar arguments. ■

Example 2.16 *The integer-part* $\lfloor \quad \rfloor : \mathbb{R} \to \mathbb{R}$ *is a quasi-isometry. This shows that a quasi-isometry could be a discontinuous map.*

Example 2.17 *Let* $\Gamma = \langle S \rangle$ *and* $\Gamma' = \langle S' \rangle$ *be groups equipped with their word metrics. Assume that* $|\Gamma'| < +\infty$*. Then, the inclusion* $i : \Gamma \hookrightarrow \Gamma \times \Gamma'$ *into the (word-metric) direct product is a quasi-isometry.*

Example 2.18 *Let* (X, d) *be a bounded metric space. Then, obviously,* X *is quasi-isometric to the one-point space. As an application, consider the groups* $\mathbb{Z} = \langle \mathbb{Z} \setminus \{0\} \rangle$ *and* $\mathbb{Q} = \langle \mathbb{Q} \setminus \{0\} \rangle$ *endowed with the word metrics corresponding to the specified sets of generators. These are bounded, hence quasi-isometric, metric spaces thus showing that being finitely generated is not preserved under quasi isometries.*

Problem 2.19 *Let* $\Gamma_1 = \langle S_1 \rangle$ *and* $\Gamma_2 = \langle S_2 \rangle$*. Suppose* (Γ_1, d_{S_1}) *and* (Γ_2, d_{S_2}) *are quasi-isometric. Is it true that* $|S_1| < +\infty$ *implies that* Γ_2 *is finitely generated? Is it true that* $|S_1| < +\infty$ *if and only if* $|S_2| < +\infty$*?*

Example 2.20 *Consider $\mathbb{Z}^m$ with the word metric corresponding to the finite set of generators $\{(1,...,0),...,(0,...,1)\}$ and let $\mathbb{R}^m$ be equipped with the Euclidean metric. Then, the inclusion $i:\mathbb{Z}^m \hookrightarrow \mathbb{R}^m$ is a quasi isometry. Note that $\mathbb{Z}^m$ acts properly by isometries on $\mathbb{R}^m$ with compact orbit space $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$. As we shall see momentarily, this is deeply related to the fact that $\mathbb{Z}^m$ is finitely generated.*

Someone refers to the next theorem as **the fundamental observation/lemma in geometric group theory**.

Theorem 2.21 (Švarc, Milnor) *Let $(M,\langle,\rangle)$ be a complete Riemannian manifold and let $\Gamma \subset \mathcal{I}(M)$ be a discrete group acting properly and freely on M. If the orbit space M/Γ is a compact manifold, then:*

(A) Γ is finitely generated.

(B) For any fixed $x_0 \in M$, the map $\phi:\Gamma \to M$ defined by $\phi(\gamma) = \gamma x_0$ is a quasi-isometry.

Accordingly, robust[3] metric properties of M can be read directly into the metric geometry of the discrete metric space Γ.

Remark 2.22 *The assumption that the action is free is not needed. Hence one could consider singular orbit spaces. Think for instance of the action of $\mathbb{Z}_k$ on $\mathbb{R}^2$ by rotations with center 0. The action is proper (because the group is finite) but the orbit-space is a cone and the projection is not even a topological covering.*

Actually, for the conclusion of Theorem 2.21 to hold, $(M,\langle,\rangle)$ can be replaced by a so called "proper and geodesic" metric space (X,d). See Remark 2.31 below for the relevant definitions.

In conclusion, from the viewpoint of geometric group theory, since one is primarily interested in groups, it is very important to relax the hypotheses on the action and on the metric space as much as possible. However, we are mainly interested in normal Riemannian coverings.

Proof. Let $P: M \to M/\Gamma$ be the Riemannian covering projection. Since M/Γ is compact, $D = \mathrm{diam}(M/\Gamma) < +\infty$. Choose a base point $x_0 \in M$ and consider the compact geodesic ball $B = \overline{B_D(x_0)} \subset M$. Observe that the family $\{\gamma B : \gamma \in \Gamma\}$ is a covering of M. Indeed, for every $x \in M$, $d_{M/\Gamma}(\Gamma x, \Gamma x_0) \leq D$. Since

$$d_{M/\Gamma}(\Gamma x, \Gamma x_0) = \mathrm{dist}_M(\Gamma x, \Gamma x_0) = \min\{d_M(x, \gamma x_0) : \gamma \in \Gamma\},$$

[3] i.e. invariant under quasi-isometries.

it follows that there exists $\gamma \in \Gamma$ such that $d_M(x, \gamma x_0) \le D$ and this implies that $x \in \gamma B$. Let

$$S = \{\gamma \in \Gamma : \gamma \neq 1 \text{ and } B \cap \gamma B \neq \emptyset\}.$$

Observe that $S = S^{-1}$ and S is a finite set because the action is proper.

(A) We show that $\Gamma = \langle S \rangle$. To this end, we proceed in two steps.

Step 1. Let

$$r = \inf\{\mathrm{dist}_M(B, \gamma B) : \gamma \notin S \cup \{1\}\}.$$

Then, $r > 0$. Indeed, suppose by contradiction that $r = 0$. Then, we find sequences $x_n \in B$, $\gamma_n y_n \in \gamma_n B$ such that $d_M(x_n, \gamma_n y_n) < \varepsilon$, for every $n >> 1$. Since B is compact, up to passing to a subsequence, $x_n \to x \in B$ and $y_n \to y \in B$. Note also that

$$d_M(\gamma_n y_n, x_0) \le d_M(\gamma_n y_n, x_n) + d_M(x_n, x_0) < D + \varepsilon.$$

Therefore, if we consider the compact ball $B' = \overline{B_{D+\varepsilon}(x_0)}$, we have, for $n >> 1$,

$$\emptyset \neq \gamma_n B \cap B' \subset \gamma_n B' \cap B'.$$

Since the action is proper, we find a constant subsequence

$$\gamma_{n_k} = \gamma \notin S \cup \{1\}. \tag{2.1}$$

It follows that $d_M(x, \gamma y) < \varepsilon$ for every $\varepsilon > 0$ proving that $x = \gamma y$. As a consequence, $x = \gamma y \in B \cap \gamma B \neq \emptyset$ which contradicts (2.1).

Step 2. We now show that $\Gamma = \langle S \rangle$. Fix $\overline{\gamma} \in \Gamma \backslash S$, $\overline{\gamma} \neq 1$. Then, by definition of S, $B \cap \overline{\gamma} B = \emptyset$ so that $d(x_0, \overline{\gamma} x_0) \ge 2D > D$. Let $k \in \mathbb{N} \backslash \{0\}$ be such that

$$D + (k-1) r \le d_M(x_0, \overline{\gamma} x_0) < D + kr. \tag{2.2}$$

We consider a minimizing geodesic $\sigma : [0, L] \to M$ from x_0 to $\overline{\gamma} x_0$ and we fix a partition $0 = t_0 < t_1 < ... < t_k = L$ such that, having set $x_j = \sigma(t_j)$, it holds

$$d_M(x_0, x_1) < D, \quad \text{and} \quad d_M(x_j, x_{j+1}) < r, \tag{2.3}$$

for every $j = 1, ..., k-1$. Since $\{\gamma B : \gamma \in \Gamma\}$ cover M, we find $\gamma_0 = 1, \gamma_1, ..., \gamma_k = \overline{\gamma}$ such that $x_j \in \gamma_j B$, for every $j = 0, ..., k-1$. By (2.3) and the definition of r it follows that

$$\gamma_j B \cap \gamma_{j+1} B \neq \emptyset,$$

for every $j = 0, ..., k-1$. Whence, recalling the definition of S, we deduce that

$$s_{j+1} = \gamma_j^{-1}\gamma_{j+1} \in S,$$

for every $j = 0, ..., k-1$. On noting that

$$\begin{aligned}\overline{\gamma} &= \gamma_k \\ &= \left(\gamma_0^{-1}\gamma_1\right)\left(\gamma_1^{-1}\gamma_2\right)\cdots\left(\gamma_{k-1}^{-1}\gamma_k\right) \\ &= s_1 \cdots s_k,\end{aligned}$$

we have thus proved that $\Gamma = \langle S \rangle$. Note that, in fact,

$$\ell_S(\overline{\gamma}) \leq k. \tag{2.4}$$

(B) It remains to prove that the map $\phi(\gamma) = \gamma x_0$ is a quasi isometry. To this aim, note that by (2.2), (2.4) and the definition of ϕ,

$$D + (d_S(1,\gamma) - 1)\, r \leq d_M(\phi(1), \phi(\gamma)),$$

for every $\gamma \notin S$. On the other hand, if $\gamma \in S$, it holds

$$\lambda d_S(1,\gamma) = \lambda \leq d_M(\phi(1), \phi(\gamma)),$$

where $\lambda = \min\{d_M(x_0, \gamma x_0) : \gamma \in S\}$. Summarizing,

$$\min\{\lambda, r\}\, d_S(1,\gamma) - r \leq d_M(\phi(1), \phi(\gamma)), \tag{2.5}$$

for every $\gamma \in \Gamma$. In order to get an upper estimate of $d_M(\phi(1), \phi(\gamma))$, write $\gamma = s_1 \cdots s_h$ with $\ell_S(\gamma) = h$ and $s_j \in S = S^{-1}$. Then, by definition of S,

$$\begin{cases} B \cap s_1 B \neq \emptyset, \\ s_1 B \cap s_1 s_2 B \neq \emptyset, \\ \cdots \\ s_1 \cdots s_{h-1} B \cap \gamma B \neq \emptyset. \end{cases}$$

For every $j = 1, ..., h-1$ we choose a point $x_j \in s_1 \cdots s_j B \cap s_1 \cdots s_{j+1} B$. Since $\operatorname{diam} B = 2D$ we have $d_M(x_j, x_{j+1}) \leq 2D$. Therefore, using the triangle inequality we deduce

$$d_M(x_0, \gamma x_0) \leq \sum_{j=0}^{h-1} d_M(x_j, x_{j+1}) \leq h2D = 2D\ell_S(\gamma). \tag{2.6}$$

Putting together (2.5) and (2.6) gives

$$\min\{\lambda, r\} d_S(1,\gamma) - r \leq d_M(\phi(1), \phi(\gamma)) \leq 2Dd_S(1,\gamma).$$

Since $d_M(\phi(\gamma_1), \phi(\gamma_2)) = d_M\left(\phi(1), \phi\left(\gamma_1^{-1}\gamma_2\right)\right)$ and, similarly, $d_S(\gamma_1,\gamma_2) = d_S\left(1, \gamma_1^{-1}\gamma_2\right)$ we conclude that ϕ is a quasi-isometric embedding. Finally, let $x \in M$. Since $\{\gamma B : \gamma \in \Gamma\}$ is a covering of M, we find $\gamma \in \Gamma$ such that $x \in \gamma B$.

$$d_M(x, \phi(\gamma)) = d_M(x, \gamma x_0) \leq D.$$

This proves that the quasi-isometric embedding ϕ is a quasi-isometry. ■

2.4 Cayley graphs and growth function

In this last subsection, we set the basic definitions and analyze some properties concerning the growth rate of finitely generated groups. In order to better visualize this concept it is perhaps useful to introduce the (metric) Cayley graph of the group. First, we set the combinatorial definition.

Definition 2.23 *Let $\Gamma = \langle S \rangle$ be a finitely generated group. The* ***Cayley graph*** *$\mathcal{G}(\Gamma, S)$ is a (non-oriented) graph*[4] *with set of vertexes V, set of edges E and (injective) end-points map $P : E \to \mathcal{P}_2(V)$ where:*

(a) $V = \Gamma$;

(b) $e \in E$ if and only if $P(e) = \{\gamma_1, \gamma_2\}$ and $d_S(\gamma_1, \gamma_2) = 1$.

Note that condition (b) means that e is an edge of $\mathcal{G}(\Gamma, S)$ precisely when there exists a generator $s \in S \cup S^{-1}$ such that the end-points $P(e) = \{\gamma_1, \gamma_2\}$ satisfy $\gamma_1 = \gamma_2 s$. Therefore, in order to construct $\mathcal{G}(\Gamma, S)$ one starts with the unit $1 \in \Gamma$, then connect the vertexes $S \cup S^{-1}$, next connect subsequently the new vertexes obtained by multiplying on the right each of the already given vertexes by an element of $S \cup S^{-1}$ and so on. Clearly, for Abelian groups, the side of multiplication is irrelevant. Note also that the **degree** (of incidence) of $\mathcal{G}(\Gamma, S)$ is constantly equal to d, where

$$d = |\{\gamma \in \Gamma : d_S(\gamma, 1) = 1\}| = \left|\overline{j}\left(S \cup S^{-1}\right) \setminus \{1\}\right|,$$

and $\overline{j} : F(S) \to \Gamma$ is the surjective homomorphism introduced in Theorem 2.4.

[4] i.e. a 1-dimensional simplicial complex. Actually one could consider 1-dimensional CW complexes thus obtaining multi-graphs with loops. However, by definition, the Cayley graph is simplicial.

Finally, note that, by construction, the Cayley graph is connected and it has neither multi-edges (more than two edges with the same vertexes) nor loop-edges (an edge with coincident vertexes).

Example 2.24 *The Cayley graph of* $\mathbb{Z}_m = \langle S|R\rangle$ *with* $S = \{a\}$ *and* $R = \{a^m\}$.

Example 2.25 *The Cayley graph of the free product* $\mathbb{Z}_2 * \mathbb{Z}_3 = \langle S|R\rangle$ *with* $S = \{a, b\}$ *and* $R = \{a^2, b^3\}$.

Example 2.26 *The Cayley graph of the free group* $\mathbb{Z} = \langle\{1\}\rangle$ *and that of the non-free group* $\mathbb{Z} = \langle\{2, 5\}\rangle$.

Example 2.27 *The Cayley graph of the free group* $F(m) = \mathbb{Z} * \cdots * \mathbb{Z}$ *with free set of generators* $S = \{x_1, ..., x_m\}$.

Example 2.28 *The Cayley graph of the "free-Abelian" group* $\mathbb{Z}^m$ *generated by* $\{(1, ..., 0), ..., (0, ..., 1)\}$.

The Cayley graph of a finitely generated group has a naturally defined distance which makes it a proper, geodesic metric space. Furthermore, the original group acts by isometries on it. A formal definition of abstract **metric (multi-)graphs** can be found in [5], whereas we will be quite informal.

Definition 2.29 *Let* $\mathcal{G}(\Gamma, S)$ *be the Cayley graph of the finitely generated group* $\Gamma = \langle S\rangle$.
Define a distance function on each edge of $\mathcal{G}(\Gamma, S)$ *by requiring that: (i)* $e \in E$ *is isometric to the unit interval* $[0, 1]$*; (ii) the group* Γ *acts by isometries on each edge* $e \approx [0, 1]$.

A path $l : [0, 1] \to \mathcal{G}(\Gamma, S)$ *is said to be* ***piecewise linear*** *if there exist a partition* $0 = t_0 < t_1 < ... < t_{m-1} < t_m = 1$ *of* $[0, 1]$ *and (adjacent) edges* $e_0, ..., e_{m-1}$ *such that* $l|_{[t_j, t_{j+1}]} : [t_j, t_{j+1}] \to e_j \approx [0, 1]$ *is affine, for* $j = 0, ..., m-1$. *The* ***length*** *of the piecewise linear path* l *is defined as*

$$\ell_{\mathcal{G}}(l) = \sum_{j=0}^{m-1} |l(t_{j+1}) - l(t_j)|.$$

Finally one defines the ***(length) distance*** *between two points* $x, y \in \mathcal{G}(\Gamma, S)$ *by the formula*

$$d_{\mathcal{G}}(x, y) = \inf\{\ell_{\mathcal{G}}(l) : l \text{ piecewise linear from } x \text{ to } y\}.$$

In the following result we collect some basic properties of the resulting metric space.

Proposition 2.30 *Let $\Gamma = \langle S \rangle$ be a finitely generated group. Consider the metric Cayley graph $(\mathcal{G}(\Gamma, S), d_{\mathcal{G}})$. Then:*

*(a) $\mathcal{G}(\Gamma, S)$ is a **geodesic space**, namely, for every $x, y \in \mathcal{G}(\Gamma, S)$ there exists a piecewise linear path $l : [0,1] \to \mathcal{G}(\Gamma, S)$ from x to y such that $d_{\mathcal{G}}(x, y) = \ell_{\mathcal{G}}(l)$.*

*(b) $\mathcal{G}(\Gamma, S)$ is a **proper space**, namely, each closed ball is compact.*

(c) The group Γ acts (transitively) by isometries on $\mathcal{G}(\Gamma, S)$.

(d) The natural inclusion $i : (\Gamma, d_S) \hookrightarrow (\mathcal{G}(\Gamma, S), d_{\mathcal{G}})$ is an isometric embedding and a quasi-isometry.

Proof. (a) Assume $x \in e$ and $y \in e'$. Then, we can find a minimum set of adjacent edges $e_0 = e, e_1, ..., e_m = e'$ with $P(e_j) = \{\gamma_j, \gamma_{j+1}\}$, $j = 0, ..., m-1$. Consider the piecewise linear path $l : [0, m+1] \to e_0 \cup ... \cup e_m$ from γ_0 to γ_m such that l has unit speed on each edge. Up to a reparametrization on $[0,1]$, a portion of l realizes the distance $d_{\mathcal{G}}(x, y)$.

(b) This follows from the fact that $\mathcal{G}(\Gamma, S)$ has a finite (uniform) vertex degree.

(c) By construction.

(d) By the very definitions of $\mathcal{G}(\Gamma, S)$ and $d_{\mathcal{G}}$, the inclusion is an isometric embedding. On the other hand, for each point $x \in \mathcal{G}(\Gamma, S)$ there exists some vertex $\gamma \in \Gamma = V$ such that $d_{\mathcal{G}}(x, \gamma) \leq 1/2$. ■

Remark 2.31 *According to the terminology introduced by Gromov, a metric space whose distance function is obtained via the infimum of the lengths of continuous paths connecting two given points is named a **length metric space**. In this general setting, the length space (X, d) is said to be a **geodesic space** if the (length) distance between two points is realized by some continuous path connecting them. For a complete, locally compact, length metric space (X, d) we have a **Hopf-Rinow type theorem** stating that (X, d) is geodesic and **proper**, i.e., closed metric balls are compact; see [16] and e.g. [5].*

With this concrete picture in mind, we now introduce the concept of growth of a finitely generated group.

Definition 2.32 *Let* Γ *be a finitely generated group with finite set of generators* S*. The associated* ***growth function*** $\beta_{\Gamma,S}(r) : [0,+\infty) \to [0,+\infty)$ *is the non-decreasing function defined by*

$$\beta_{\Gamma,S}(r) = |\{\gamma \in \Gamma : \ell_S(\gamma) \leq r\}| = \left|\overline{\mathcal{B}_r(1)}\right|,$$

where $\overline{\mathcal{B}_r(\gamma)}$ *denotes the closed ball in* (Γ, d_S) *centered at* $\gamma \in \Gamma$ *and of radius* $r > 0$*. Obviously,* $\beta_{\Gamma,S}(r) = \beta_{\Gamma,S}(\lfloor r \rfloor)$ *where, we recall,* $\lfloor r \rfloor \in \mathbb{N}$ *is the integer part of* r*.*

Clearly, one can visualize $\beta_{\Gamma,S}(r)$ as the number of vertexes in the closed ball of $(\mathcal{G}(\Gamma,S), d_{\mathcal{G}})$ centered at 1 and of radius $r > 0$. In this way it is quite simple to compute the growth function or the growth rate for the Examples 2.24–2.28 above.

Example 2.33 *Let* $\mathbb{Z}_m = \langle S|R \rangle$ *with* $S = \{a\}$, *and* $R = \{a^m\}$*. Then, for every* $r \geq 0$*,*

$$\beta_{\mathbb{Z}_m,S}(r) = \min\{2\lfloor r \rfloor + 1, m\}.$$

Example 2.34 *Let* $F(m) = \mathbb{Z} * \cdots * \mathbb{Z} = \langle S|R \rangle$ *with* $S = \{x_1, ..., x_m\}$ *and* $R = \emptyset$*. Then, for every* $r \geq 1$*,*

$$\begin{aligned}\beta_{F(m),S}(r) &= 1 + 2m \sum_{j=1}^{\lfloor r \rfloor} (2m-1)^{j-1} \\ &= -\frac{1}{m-1} + \frac{m}{m-1}(2m-1)^{\lfloor r \rfloor}.\end{aligned}$$

Example 2.35 *Let* $\mathbb{Z}_2 * \mathbb{Z}_3 = \langle S|R \rangle$ *with* $S = \{a, b\}$ *and* $R = \left\{a^2, b^3\right\}$*. Then, we have the estimate*

$$\beta_{\mathbb{Z}_2 * \mathbb{Z}_3, S}(r) \geq 2^{\lfloor r \rfloor - 1} - 1,$$

for every $r \geq 2$*. Actually, one can easily exhibit a recursive formula for the exact value of* $\beta_{\mathbb{Z}_2 * \mathbb{Z}_3, S}(k)$, $k \in \mathbb{N}$*.*

Example 2.36 *Let* $\mathbb{Z}^2$ *be generated by* $S = \{(1,0), (0,1)\}$*. Then, for every* $r \geq 0$*,*

$$\begin{aligned}\beta_{\mathbb{Z}^2,S}(r) &= 2 \sum_{j=0}^{\lfloor r \rfloor - 1} (2j+1) + 2\lfloor r \rfloor + 1 \\ &= 2\lfloor r \rfloor^2 + 2\lfloor r \rfloor + 1.\end{aligned}$$

Similarly, for $\mathbb{Z}^m$ with the set of generators $S = \{(1, ..., 0), ... (0, ..., 1)\}$ one has

$$\beta_{\mathbb{Z}^m,S}(r) \asymp \lfloor r \rfloor^m .$$

Although one could compute the exact value of $\beta_{\mathbb{Z}^m,S}(r)$, note that its asymptotic behavior is an immediate consequence of the fact that $\mathbb{Z}^m$ acts properly on $\mathbb{R}^m$ with compact orbit space $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$. Therefore we can apply Švarc-Milnor Theorem 3.1 in Chapter 3 below.

The next lemma encloses some easy properties of the growth function.

Lemma 2.37 *Let Γ be a finitely generated group and let S be a finite set of generators. Then:*

(a) If S' is a second finite set of generators of Γ then there exists a constant $a \geq 1$ such that

$$\beta_{\Gamma,S'}\left(a^{-1}r\right) \leq \beta_{\Gamma,S}(r) \leq \beta_{\Gamma,S'}(ar).$$

(b) The growth function $\beta_{\Gamma,S}(r)$ is always dominated by the growth function of the corresponding free group $F(S)$:

$$\beta_{\Gamma,S}(r) \leq \beta_{F(S),S}(r).$$

(c) The growth function $\beta_{\Gamma,S}$ is sub-multiplicative in the sense that,

$$\beta_{\Gamma,S}(k+h) \leq \beta_{\Gamma,S}(k)\,\beta_{\Gamma,S}(h),$$

for every $k, h \in \mathbb{N}$. In particular, for every $r \geq 1$,

$$\beta_{\Gamma,S}(r) \leq b^r,$$

with $b = \beta_{\Gamma,S}(1)$. In fact,

$$\beta_{\Gamma,S}(k)^{\frac{1}{k}} \to \beta < +\infty, \tag{2.7}$$

as $k \to +\infty$.

(d) If Γ is infinite, then $\beta_{\Gamma,S}(k)$ is strictly increasing in $k \in \mathbb{N}$ and, obviously, $\beta_{\Gamma,S}(k) \to +\infty$ as $k \to +\infty$.

Proof. To prove (a), simply recall from Lemma 2.13 that (Γ, d_S) and $(\Gamma, d_{S'})$ are bilipschitz equivalent. Therefore the corresponding metric balls satisfy $\overline{\mathcal{B}'_{r/\lambda}(1)} \subset \overline{\mathcal{B}_r(1)} \subset \overline{\mathcal{B}'_{r\lambda}(1)}$, for some constant $\lambda \geq 1$.

Property (b) follows from the fact that by adding a relation decreases the vertex degree in the corresponding Cayley graph.

As for property (c), observe that

$$\overline{\mathcal{B}_{r+s}(1)} = \overline{\mathcal{B}_r(1)} \cdot \overline{\mathcal{B}_s(1)}.$$

Indeed, the inclusion $\supseteq$ follows from the subadditivity of the word length; see (b) of Lemma 2.12. Thus, for every $\gamma_1 \in \overline{\mathcal{B}_r(1)}$ and $\gamma_2 \in \overline{\mathcal{B}_s(1)}$ it holds $\gamma_1\gamma_2 \in \overline{\mathcal{B}_{r+s}(1)}$. On the other hand, the inclusion $\subseteq$ is a consequence of the fact that, using generators, every $\gamma \in \overline{\mathcal{B}_{r+s}(1)}$ can be decomposed as a product of some $\gamma_1 \in \overline{\mathcal{B}_r(1)}$ and $\gamma_2 \in \overline{\mathcal{B}_s(1)}$, possibly $\gamma_j = 1$. Since

$$\left|\overline{\mathcal{B}_r(1)} \cdot \overline{\mathcal{B}_s(1)}\right| \leq \left|\overline{\mathcal{B}_r(1)} \times \overline{\mathcal{B}_s(1)}\right| = \left|\overline{\mathcal{B}_r(1)}\right| \left|\overline{\mathcal{B}_s(1)}\right|,$$

the asserted property follows. To prove (2.7) it suffices to show that

$$\limsup_{k_1 \to +\infty} \beta_{\Gamma,S}(k_1)^{\frac{1}{k_1}} \leq \liminf_{k_2 \to \infty} \beta_{\Gamma,S}(k_2)^{\frac{1}{k_2}}. \tag{2.8}$$

Fix k_1, k_2 and let $k = \lfloor k_1/k_2 \rfloor + 1$. Then, by monotonicity and by the submultiplicative property,

$$\beta_{\Gamma,S}(k_1) \leq \beta_{\Gamma,S}(k_2 k) \leq \beta_{\Gamma,S}(k_2)^k \leq \beta_{\Gamma,S}(k_2)^{1+k_1/k_2},$$

which implies

$$\beta_{\Gamma,S}(k_1)^{\frac{1}{k_1}} \leq \beta_{\Gamma,S}(k_2)^{\frac{1}{k_2}} \beta_{\Gamma,S}(k_2)^{\frac{1}{k_1}}.$$

Letting $k_1 \to +\infty$ we deduce

$$\limsup_{k_1 \to +\infty} \beta_{\Gamma,S}(k_1)^{\frac{1}{k_1}} \leq \beta_{\Gamma,S}(k_2)^{\frac{1}{k_2}}$$

proving (2.8).

Property (d) is obvious. ■

We are going to extend the conclusion in (a) of Lemma 2.37 to the case of quasi-isometric, finitely generated groups. To this purpose it is useful to set the next

Definition 2.38 *Suppose we are given non-decreasing functions $\alpha, \beta : [0, +\infty) \to [0, +\infty)$. These are generically called* ***growth functions****. We say that α (weakly)*[5] ***dominates*** *β if*

$$\beta(t) \leq A\alpha(At + B) + B,$$

for every $t \geq 0$ and for some constants $A > 0, B \geq 0$. In this case we write $\beta \preccurlyeq \alpha$. In case $\alpha \preccurlyeq \beta$ and $\beta \preccurlyeq \alpha$ we say that the growth functions α, β are (weakly) ***equivalent*** *and we write $\alpha \approx \beta$.*

Obviously, the (weak) equivalence of growth functions is an equivalence relation. Because of Lemma 2.37 (a), the equivalence class of the growth function associated to a finitely generated group does not depend on the finite set of generators. This class is named the **growth type** of the group. Accordingly, given $\Gamma = \langle S \rangle$ with finite set of generators S we shall often write $\beta_\Gamma(t)$ instead of $\beta_{\Gamma,S}(t)$.

Definition 2.39 *Let Γ be a finitely generated group. We say that Γ has*

(a) ***polynomial growth of degree*** *d if $\beta_\Gamma(r) \approx r^d$;*

(b) ***polynomial growth of degree*** *$\leq d$ if $\beta_\Gamma(r) \preccurlyeq r^d$;*

(c) ***exponential growth*** *if $e^r \preccurlyeq \beta_\Gamma(r)$*

Remark 2.40 *Note that, by Lemma 2.37 (c), the requirement in (c) of the previous definition is equivalent to $\beta_\Gamma(r) \approx e^r$. Note also that one has intermediate growth types between polynomial and exponential. Groups of intermediate growths were discovered by Grigorchuk answering a question posed by Milnor, see [21]. For a recent beginner's guide, see [14].*

The next Theorem states that the growth type of a finitely generated group is a quasi-isometry invariant. As an immediate application we have that the free group $F(m) = \mathbb{Z} * \cdots * \mathbb{Z}$ is not quasi-isometric to a free Abelian group $\mathbb{Z}^n$. This latter, in turn, cannot be quasi-isometric to $\mathbb{Z}^k$ for $k \neq n$. The group $\mathbb{Z}^k$ is not quasi-isometric to $\mathbb{Z}_2 * \mathbb{Z}_3$ and so on. Summarizing, the growth type enables us to distinguish finitely generated groups up to quasi-isometries. The growth type is just one of the quasi isometric invariants. Another important invariant is the number of ends of

[5]opposed to strong domination which is defined by $\beta(t) \leq \alpha(\lambda t)$, for every $t \geq 0$ and for some $\lambda > 0$. It can be shown that, for infinite, finitely generated groups weak and strong dominations are equivalent; [21].

$\Gamma = \langle S \rangle$ which is defined as the number of the ends of the corresponding Cayley graph $(\mathcal{G}(\Gamma, S), d_{\mathcal{G}})$; [5]. Further invariants are discussed e.g. in [15], [5], [21].

Theorem 2.41 *Let $\Gamma = \langle S \rangle$ and $\Gamma' = \langle S' \rangle$ be finitely generated groups. Assume that there is a quasi-isometric embedding $\phi : (\Gamma, d_S) \to (\Gamma', d'_{S'})$. Then,*

$$\beta_\Gamma \preccurlyeq \beta_{\Gamma'}.$$

In particular, if the quasi-isometric embedding ϕ is a quasi-isometry then,

$$\beta_\Gamma \approx \beta_{\Gamma'}.$$

Proof. By assumption, there exist $\lambda \geq 1$ and $L \geq 0$ such that

$$\lambda^{-1} d_S(\gamma_1, \gamma_2) - L \leq d'_{S'}(\phi(\gamma_1), \phi(\gamma_2)) \leq \lambda d_S(\gamma_1, \gamma_2) + L. \tag{2.9}$$

In particular, from the second inequality in (2.9) we have

$$\phi\left(\overline{\mathcal{B}_R(1)}\right) \subset \overline{\mathcal{B}'_{\lambda R + L}(\phi(1))},$$

where, we recall, $\mathcal{B}$ and $\mathcal{B}'$ denote the metric balls of (Γ, d_S) and $(\Gamma', d_{S'})$, respectively. Therefore

$$\left|\overline{\mathcal{B}_R(1)}\right| \leq \left|\phi^{-1}\left(\overline{\mathcal{B}'_{\lambda R + L}(\phi(1))}\right)\right|. \tag{2.10}$$

On the other hand, the first inequality in (2.9) implies that, for every $\gamma_1, \gamma_2 \in \Gamma$ with $\phi(\gamma_1) = \gamma' = \phi(\gamma_2)$, it holds

$$d_S(\gamma_1, \gamma_2) \leq \lambda L$$

proving that

$$\phi^{-1}(\gamma') \subseteq \overline{\mathcal{B}_{\lambda L}(\gamma_1)}.$$

Therefore,

$$\left|\phi^{-1}(\gamma')\right| \leq \left|\overline{\mathcal{B}_{\lambda L}(\gamma_1)}\right| = \left|\overline{\mathcal{B}_{\lambda L}(1)}\right|$$

for every $\gamma' \in \Gamma'$. Using this information into (2.10) we conclude

$$\left|\overline{\mathcal{B}_R(1)}\right| \leq C \left|\overline{\mathcal{B}'_{\lambda R + L}(\phi(1))}\right| = C \left|\overline{\mathcal{B}'_{\lambda R + L}(1')}\right|,$$

with $C = \left|\overline{\mathcal{B}_{\lambda L}(1)}\right|$. ■

In view of future purposes, see Example 3.8 in Chapter 3, we now apply Theorem 2.41 and the fundamental lemma to deduce that the growth type of a group is preserved by finite index extensions.

Corollary 2.42 *Let* Γ *be a finitely generated group that contains a subgroup* $\Gamma' \lhd \Gamma$ *such that* $|\Gamma/\Gamma'| < +\infty$. *Then,* Γ' *is finitely generated and quasi-isometric to* Γ. *In particular,* $\beta_{\Gamma}(r) \approx \beta_{\Gamma'}(r)$.

Proof. Consider the Cayley graph $\mathcal{G}(\Gamma, S)$ of Γ with respect to a finite set of generators S. Then Γ' acts properly (but in general not freely[6]) on $\mathcal{G}(\Gamma, S)$. Since Γ' has finite index in Γ it follows that $\mathcal{G}(\Gamma, S)/\Gamma'$ is compact. Therefore, the fundamental Lemma in the general context of geodesic spaces implies that Γ' is finitely generated and quasi-isometric to $\mathcal{G}(\Gamma, S)$. On the other hand, $\mathcal{G}(\Gamma, S)$ is quasi-isometric to Γ, therefore Γ and Γ' are quasi-isometric. From Theorem 2.41 it follows that these groups have the same growth type. ■

[6]think for instance to $\Gamma = \mathbb{Z}_2 = \Gamma'$. The mid-point is left fixed.

Chapter 3

Fundamental group of a compact manifold with $Ric \geq 0$

We begin this section by presenting Švarc-Milnor result(s) on the growth type of the fundamental group of a compact manifold of non-negative Ricci curvature. This result will follow immediately from the next general and fundamental theorem which relates the volume growth of geodesic balls of a complete Riemannian manifold with the growth type of a discrete group acting freely, properly and co-compactly[1] on it. In a sense, it completes the picture drawn in Theorem 2.21 above.

Theorem 3.1 (Švarc-Milnor) *Let Γ be a (finitely generated) discrete group acting freely[2] and properly on a complete Riemannian manifold $(M, \langle , \rangle)$. Assume that the orbit space M/Γ is compact. Then, having fixed an origin $o \in M$, the growth functions $v_M(r) = \mathrm{Vol} B_r(o) = \mathrm{Vol}\overline{B_r(o)}$ of M and $\beta_\Gamma(r)$ of Γ satisfy $v_M(r) \approx \beta_\Gamma(r)$.*

Obviously, the growth type of v_M does not depend on the choice of the origin $o \in M$. Indeed, if $o' \in M$ is a second origin and $\delta = d(o, o')$ then, by the triangle inequality, we have the inclusions

$$B_{R-\delta}(o) \subseteq B_R(o') \subseteq B_{R+\delta}(o),$$

[1] i.e. with compact orbit space.

[2] actually, up to minor changes in the proof, the same conclusion holds even if the action is not free and, hence, M/Γ is not a manifold.

for every $R > \delta$.

Proof. First of all, we recall from Theorem 2.21 that Γ is generated by the finite set

$$S = S^{-1} = \left\{ \gamma \in \Gamma : \overline{B_D(x_0)} \cap \gamma \overline{B_D(x_0)} \neq \emptyset \right\},$$

where $D = \operatorname{diam}(M)$. Furthermore, $\left\{ \gamma \overline{B_D(x_0)} : \gamma \in \Gamma \right\}$ cover M. Finally, having fixed a point $x_0 \in M$, the (injective) map $\phi : \Gamma \to M$ such that $\phi(\gamma) = \gamma x_0$ is a quasi-isometry. In particular, there exist constants $\lambda \geq 1$ and $L \geq 0$ such that

$$d(x_0, \gamma x_0) \geq \frac{1}{\lambda} d_S(\gamma, 1) - L. \tag{3.1}$$

Part 1. We show that $v_M(r) \preccurlyeq \beta_\Gamma(r)$. Let

$$\mu = \max\{d(x_0, s x_0) : s \in S\}.$$

Take any $\gamma \in \overline{\mathcal{B}_r(1)} \subset (\Gamma, d_S)$ and set $\gamma = s_1 \cdots s_k$ with $s_j \in S$ and $k = \ell_S(\gamma) \leq r$. Then, using the triangle inequality we deduce

$$\begin{aligned} d(x_0, \gamma x_0) &\leq d(x_0, s_1 x_0) + d(s_1 x_0, s_1 s_2 x_0) + \cdots + d(s_1 \cdots s_{k-1} x_0, \gamma x_0) \\ &= \sum_{j=1}^{k} d(x_0, s_j x_0) \leq k\mu \leq r\mu, \end{aligned}$$

thus proving that

$$\gamma \in \overline{\mathcal{B}_r(1)} \Longrightarrow \gamma x_0 \in \overline{B_{r\mu}(x_0)}. \tag{3.2}$$

Now, since the action is free and proper, hence properly discontinuous, we can find $0 < \varepsilon << 1$ such that

$$\gamma' \overline{B_\varepsilon(x_0)} \cap \overline{B_\varepsilon(x_0)} = \emptyset, \ \forall \gamma' \in \Gamma.$$

Since, by (3.2),

$$\cup_{\gamma \in \overline{\mathcal{B}_r(1)}} \gamma \overline{B_\varepsilon(x_0)} \subseteq \overline{B_{r\mu+\varepsilon}(x_0)},$$

taking volumes we conclude

$$\begin{aligned} \left| \overline{\mathcal{B}_r(1)} \right| \operatorname{Vol}\left(\overline{B_\varepsilon(x_0)} \right) &= \sum_{\gamma \in \overline{\mathcal{B}_r(1)}} \operatorname{Vol}\left(\gamma \overline{B_\varepsilon(x_0)} \right) \\ &= \operatorname{Vol}\left(\cup_{\gamma \in \overline{\mathcal{B}_r(1)}} \gamma \overline{B_\varepsilon(x_0)} \right) \\ &\leq \operatorname{Vol}\left(\overline{B_{r\mu+\varepsilon}(x_0)} \right), \end{aligned}$$

that is,

$$\beta_{\Gamma}(r) \leq C v_M(\mu r + \varepsilon),$$

with

$$C = \mathrm{Vol}\left(\overline{B_{\varepsilon}(x_0)}\right)^{-1}.$$

Part 2. We now prove that $v_M(r) \preccurlyeq \beta_{\Gamma}(r)$. Let us consider the ball $\overline{B_R(x_0)}$, $R > 0$. Since $\left\{\gamma \overline{B_D(x_0)} : \gamma \in \Gamma\right\}$ cover M, we find $\gamma_1, ..., \gamma_t \in \Gamma$ such that

$$\overline{B_R(x_0)} \subseteq \cup_{j=1}^{t} \gamma_j \overline{B_D(x_0)}. \tag{3.3}$$

Note that we can assume $\gamma_j x_0 \in \overline{B_{R+D}(x_0)}$ for, otherwise, $\gamma_j \overline{B_D(x_0)} \cap \overline{B_R(x_0)} = \emptyset$. Using (3.1) we deduce that, for every $j = 1, ..., t$,

$$d_S(\gamma_j, 1) \leq \lambda(d(x_0, \gamma_j x_0) + L) \leq \lambda R + \lambda(D + L),$$

that is, every $\gamma_j \in \overline{\mathcal{B}_{\lambda R + \lambda(D+L)}(1)}$. Therefore,

$$t \leq \left|\overline{\mathcal{B}_{\lambda R + \lambda(D+L)}(1)}\right|$$

and taking volumes in (3.3) we conclude

$$\begin{aligned} \mathrm{Vol}\left(\overline{B_R(x_0)}\right) &\leq \sum_{j=1}^{t} \mathrm{Vol}\left(\gamma_j \overline{B_D(x_0)}\right) \\ &= t\mathrm{Vol}\left(\overline{B_D(x_0)}\right) \\ &\leq \left|\overline{\mathcal{B}_{\lambda R + \lambda(D+L)}(1)}\right| \mathrm{Vol}\left(\overline{B_D(x_0)}\right), \end{aligned}$$

that is

$$v_M(R) \leq E \beta_{\Gamma}(\lambda R + \lambda(D + L)),$$

with

$$E = \mathrm{Vol}\left(\overline{B_D(x_0)}\right)$$

This completes the proof. ■

As Part 1 in the above proof shows, in order to deduce that $\beta_{\Gamma}(r) \preccurlyeq v_M(r)$ the compactness of the orbit space is merely used to guarantee that Γ is finitely generated. Thus, one can follow step by step the above arguments to get the following

Theorem 3.2 (Švarc-Milnor) *Let Γ be a discrete group acting freely and properly on a complete Riemannian manifold $(M, \langle , \rangle)$. Then, for every finitely generated subgroup $\Gamma_1 \subset \Gamma$ it holds $\beta_{\Gamma_1}(r) \preccurlyeq v_M(r)$.*

We are now in the position to give the announced topological result.

Corollary 3.3 (Milnor) *Let $(M, \langle , \rangle)$ be a complete Riemannian manifold of dimension m. If $Ric \geq 0$ then every finitely generated subgroup of $\pi_1(M)$ has polynomial growth of degree $\leq m$. If M is compact, then the same conclusion holds for $\pi_1(M)$ itself.*

Proof. We realize $M = M'/\Gamma$ where $P : M' \to M = M'/\Gamma$ is the Riemannian universal covering and $\Gamma = \pi_1(M) \subset \mathcal{I}(M')$. Let $\Gamma_1 \subset \Gamma$ be a finitely generated subgroup. According to Theorem 3.2, $\beta_{\Gamma_1}(r) \preccurlyeq v_{M'}(r)$. On the other hand, since $Ric' \geq 0$, by Bishop-Gromov comparison we have that $v_{M'}(r) \preccurlyeq r^m$.

Finally, in case $M = M'/\Gamma$ is compact, the fundamental lemma states that $\Gamma = \pi_1(M)$ is finitely generated and the first part of the proof applies. ∎

Remark 3.4 *As it is clear from the proof, one could consider complete manifolds $(M, \langle , \rangle)$ with $Ric \geq -(m-1)c^2 > 0$ and deduce that, for every finitely generated subgroup $\Gamma_1 \subset \pi_1(M)$, it holds $\beta_{\Gamma_1}(r) \preccurlyeq e^r$. However, this conclusion is obvious and holds regardless of any curvature assumption in view of statement (c) of Lemma 2.37.*

Example 3.5 *The connected sum of tori $\mathbb{T}^m \# \mathbb{T}^m$, $m \geq 3$, does not admit any metric with $Ric \geq 0$. Indeed, its fundamental group $\pi_1(\mathbb{T}^m \# \mathbb{T}^m) \simeq \mathbb{Z}^m * \mathbb{Z}^m$ is of exponential growth. Note that, in general, if a finitely generated group Γ contains (an isomorphic image of) the free group $F(2)$ then Γ has exponential growth. Thus, for instance, if M and N are compact, m-dimensional manifolds, $m \geq 3$, whose fundamental groups contain an element of infinite period, then $M \# N$ does not support any metric with $Ric \geq 0$.*

In Chapter 5, following Anderson paper [3], we will prove that under certain (polynomial) volume growth assumptions on M and M/Γ, a domination of the form $\beta_\Gamma(r) \preccurlyeq v_M(r) / v_{M/\Gamma}(r)$ occurs for the finitely generated group Γ. Using this fact, even in case (a-priori) Γ is not assumed to be finitely generated, we shall conclude that the group is finite provided $v_M(r) / v_{M/\Gamma}(r) \leq C$. To this end, we will need to introduce some more aspects of discrete group actions on Riemannian manifolds; see Chapter 4. We also note that Corollary 3.3 led Milnor to set the following still open

Conjecture 3.6 (Milnor) *Let $(M, \langle , \rangle)$ be a complete, m-dimensional manifold with $Ric \geq 0$. Then, $\pi_1(M)$ is finitely generated and of polynomial growth of degree $\leq m$.*

The following question arises naturally.

Problem 3.7 *Milnor theorem exhibits an upper bound for the growth type of the fundamental group of the compact manifold M provided $Ric \geq 0$. Does some rigidity appear in case the fundamental group is exactly of polynomial growth of degree $m = \dim M$?*

Example 3.8 *Clearly, the fundamental group of the flat torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ realizes the asserted growth type condition. Actually, the same holds for any compact, m-dimensional, flat manifold M. Indeed, by the Hopf classification theorem, M is universally covered by $\mathbb{R}^m$ and, hence, $\Gamma = \pi_1(M) \subset \mathcal{I}(\mathbb{R}^m)$ the group of rigid motions. Since $M = \mathbb{R}^m/\Gamma$ is compact, Γ is called a* ***crystallographic group****. By the* ***Bieberbach first theorem***[3]*, [9], there exists a short exact sequence*

$$0 \to \mathbb{Z}^m \to \Gamma \to \Phi \to 1,$$

where $\mathbb{Z}^m \lhd \Gamma$ is the unique maximal, free Abelian subgroup of Γ (the translational part of Γ) and Φ is a finite group (the rotational part of Γ). Since $\Phi \simeq \Gamma/\mathbb{Z}^m$, by Corollary 2.42 we see that

$$\beta_\Gamma(r) \approx \beta_{\mathbb{Z}^m}(r) \approx r^m.$$

Note that M is a finite quotient of a flat torus $\mathbb{T}^m$ and we have the finite normal covering

$$\mathbb{T}^m \to M = \mathbb{T}^m/\Phi.$$

We are going to answer in the affirmative the above question. This requires the use of a deep structure theorem which is known in the literature as the **Cheeger-Gromoll splitting theorem**; [10]. We first recall the following

[3]the Bieberbach **second theorem** states that if two crystallographic groups are isomorphic, then the isomorphism is realized by an affine change of coordinates. Finally, the **third theorem** states that, for each dimension and up to affine change of coordinates, there are only finitely many crystallographic groups.

Definition 3.9 *Let $(M, \langle,\rangle)$ be a complete Riemannian manifold. A **ray** in M issuing from $o \in M$ is a unit speed, minimizing geodesic $\sigma : [0,+\infty) \to M$ such that $\sigma(0) = o$. In case σ is minimizing on all of $\mathbb{R}$ we say that $\sigma : \mathbb{R} \to M$ is a **line.***

Example 3.10 *Every complete manifold $(M, \langle,\rangle)$ has a ray. Actually, given a compact domain $\Omega \subset M$ let $E^1, ..., E^k$ be the unbounded connected components of $M \backslash \Omega$. Each E^j is called an **end** of M with respect of Ω. Let $o \in \Omega$. Then M has (at least) k distinct rays $\sigma^j : [0,+\infty) \to M$, $j = 1, ..., k$, issuing from o and such that $\sigma^j(t) \in E^j$, $t >> 1$. To see this, fix j and take a sequence $p_n \in E^j$ such that $p_n \to \infty$. Consider a sequence of unit speed, minimizing geodesics $\sigma_n^j(t) = \exp_o(tv_n) : [0, L_n] \to M$ such that $\sigma_n^j(0) = o$, $\sigma_n^j(L_n) = p_n$. Since $v_n \in \mathbb{S}^{m-1} \subset T_oM$, we find $v_{n_h} \to v^j \in \mathbb{S}_o^{m-1} \subset T_oM$. Therefore, the corresponding subsequence of geodesics $\sigma_{n_h}^j$ converges, uniformly on compact subsets of $[0,+\infty)$, to a geodesic $\sigma^j(t) = \exp_o(tv^j) : [0,+\infty) \to M$ such that $\sigma^j(t) \in E^j$ for $t >> 1$. Since $d(\sigma_n^j(s), \sigma_n^j(t)) = |s-t|$, taking limits as $n \to +\infty$ shows that $d(\sigma^j(s), \sigma^j(t)) = |s-t|$, proving that each σ^j is a ray. One can also verify that, for $j \neq j'$, $d(\sigma^j(t), \sigma^{j'}(t)) \to +\infty$ as $t \to +\infty$.*

Example 3.11 *In general, a complete manifold (even if Ric ≥ 0) has no lines. For instance, the paraboloid $z = x^2 + y^2$ of $\mathbb{R}^3$ is a complete surface with no-lines, with one end and with positive Gaussian curvature.*

Example 3.12 *A complete manifold $(M, \langle,\rangle)$ with at least two ends has a line. Indeed, let E^1 and E^2 be two ends of M with respect to $\overline{B_R(o)} \subset M$. Following ideas in [10], take a sequence $p_n \in E^1$ such that $L_n = d(p_n, o) \to +\infty$ and, as in Example 3.10, for each n construct a ray $\widetilde{\sigma}_n : [0,+\infty) \to M$ issuing from $v_n(0) = p_n$ and satisfying $\widetilde{\sigma}_n(t) \in E^2$, $t >> 1$. Let T_n be the first entrance time of $\widetilde{\sigma}_n(t)$ in $\overline{B_R(o)}$. Since $\widetilde{\sigma}_n(T_n) \in \partial B_R(o)$, by the triangle inequality*

$$T_n = d(p_n, \widetilde{\sigma}_n(T_n)) \geq L_n - R \to +\infty, \text{ as } n \to +\infty.$$

Define $\sigma_n(t) = \widetilde{\sigma}_n(t + T_n) : [-T_n, +\infty) \to M$ which is a unit speed, minimizing geodesic. Set $\sigma_n(t) = \exp_{\sigma_n(0)}(tv_n)$, with $v_n \in \mathbb{S}_{\sigma_n(0)}^{m-1} \subset T_{\sigma_n(0)}M$ and $\sigma_n(0) \in \partial B_R(o)$. By compactness, it follows that the exists a subsequence

$$(\sigma_{n_h}(0), v_{n_h}) \overset{TM}{\to} (p, v),$$

where $p \in \partial B_R(o)$ and $v \in \mathbb{S}_p^{m-1} \subset T_pM$. The limit geodesic $\sigma(t) = \exp_p(tv) : \mathbb{R} \to M$ is the desired line.

Theorem 3.13 (Cheeger-Gromoll) *Let $(M, \langle,\rangle)$ be a complete Riemannian manifold of dimension m and satisfying $Ric \geq 0$. If M contains a line, then M is isometric to the Riemannian product $\overline{M} \times \mathbb{R}$, where $\overline{M}$ is a complete, totally geodesic hypersurface satisfying $\overline{Ric} \geq 0$.*

In the compact setting, topological conclusions for non-negative Ricci curved manifolds rely on the next companion result.

Theorem 3.14 (Cheeger-Gromoll) *Let $(M, \langle,\rangle)$ be a complete Riemannian manifold of dimension m and satisfying $Ric \geq 0$. Then, the Riemannian universal covering space $(M', \langle,\rangle')$ splits isometrically as the Riemannian product $N \times \mathbb{R}^k$, $0 \leq k \leq m$, where N is a compact(!), simply connected manifold of non-negative Ricci curvature.*

Let us assume for the moment the validity of Theorems 3.13 and 3.14; their proofs are postponed at the end of the section. Here, we show how they can be used to answer Problem 3.7. The following result can be considered as the $Ric \geq 0$-analog of the equality case in Theorem 1.7 of Chapter 1 above.

Corollary 3.15 (Cheeger-Gromoll) *Let $(M, \langle,\rangle)$ be a compact manifold of dimension m such that $Ric \geq 0$. If $\pi_1(M)$ is of polynomial growth of degree m, then M is flat and the classification reported in Example 3.8 applies. In particular, $M = \mathbb{T}^m/\Phi$ where $\Phi \simeq \pi_1(M)/\mathbb{Z}^m$ is a finite group.*

Proof. Introduce the Riemannian universal covering $P : M' \to M$ and apply Theorem 3.14 to deduce the isometric splitting $M' = \overline{M'} \times \mathbb{R}^k$, where $\overline{M'}$ is compact. Then, the volume growth function of M' satisfies $v_{M'}(r) = v_{\overline{M'} \times \mathbb{R}^k} \approx r^k$. On the other hand, by assumption and by Theorem 3.1 we have that $v_{M'}(r) \approx r^m$. It follows that $k = m$ and $M' = \mathbb{R}^m$. ■

The rest of the section is devoted to the proof of the splitting theorems. We begin with a preliminary

Lemma 3.16 *Let $\sigma : [0, +\infty) \to M$ be a ray issuing from $\sigma(0) = o \in M$ in the complete manifold $(M, \langle,\rangle)$. For every $t \geq 0$, define $b_{\sigma(t)} : M \to \mathbb{R}$ by*

$$b_{\sigma(t)}(x) = d(x, \sigma(t)) - d(o, \sigma(t)) = d(x, \sigma(t)) - t.$$

Then:

(a) For ever $x \in M$ fixed, the function $t \mapsto b_{\sigma(t)}(x)$ is decreasing. Moreover

$$\sup_{t \geq 0} \left| b_{\sigma(t)}(x) \right| \leq d(x, o).$$

(b) For every $t \geq 0$ fixed, the function $b_{\sigma(t)}$ is Lipschitz (hence differentiable a.e.) with $\left|\nabla b_{\sigma(t)}\right| = 1$.

(c) The limit function $b_\sigma : M \to \mathbb{R}$ defined by

$$b_\sigma(x) = \lim_{t \to +\infty} b_{\sigma(t)}(x)$$

*is named the **Busemann function** with respect to σ. It is Lipschitz (hence differentiable a.e.) with $|\nabla b_\sigma| = 1$. More precisely, for almost every $x \in M$, there exists a sequence $\nabla b_{\sigma(n_k)}(x)$ converging to $\nabla b_\sigma(x)$.*

Proof. (a) Fix $x \in M$ and let $t \geq s \geq 0$. Then, by the triangle inequality we have

$$\begin{aligned} b_{\sigma(t)}(x) &= d(x, \sigma(t)) - t \\ &\leq d(x, \sigma(s)) + d(\sigma(s), \sigma(t)) - t \\ &= d(x, \sigma(s)) + (t - s) - t \\ &= b_{\sigma(s)}, \end{aligned}$$

proving that $b_{\sigma(t)}(x) \leq b_{\sigma(s)}(x)$. Moreover

$$\left|b_{\sigma(t)}(x)\right| = |d(x, \sigma(t)) - d(o, \sigma(t))| \leq d(x, o).$$

(b) The asserted properties follow immediately from the analogous properties of the distance function from a fixed origin.

(c) According to (a), the Busemann function is well defined and, by (b),

$$d(b_\sigma(x), d_\sigma(y)) = \lim_{t \to +\infty} d\left(b_{\sigma(t)}(x), d_{\sigma(t)}(y)\right) \leq d(x, y),$$

proving that b_σ is 1-Lipschitz. In particular b_σ is differentiable almost everywhere, say on a set Ω_σ of full measure in M. Let

$$\Omega = \Omega_\sigma \cap_{n \in \mathbb{N}} (M \backslash \text{cut}(\sigma(n)))$$

and observe that Ω has full measure in M. Pick $x \in \Omega$. According to (b), we have $|\nabla b_\sigma(x)| \leq 1$. For every n, let $\gamma_{x,\sigma(n)} : [0, d(x, \sigma(n))] \to M$ be a unit speed, minimizing geodesic from x to $\sigma(n)$. Set $\gamma_{x,\sigma(n)}(s) = \exp_x\left(s v_{\sigma(n)}\right)$ with $v_{\sigma(n)} \in \mathbb{S}_x^{m-1} \subset T_x M$. Then, a subsequence $v_{\sigma(n_k)}$ converges to some $v \in \mathbb{S}_x^{m-1} \subset T_x M$ and the corresponding sequence of geodesics $\gamma_{x,\sigma(n)}(s)$ converges to the ray $\gamma_x(s) = \exp_x(sv) : [0, +\infty) \to M$. Note that

$$b_{\sigma(n_k)} \circ \gamma_{x,\sigma(n_k)}(s) = b_{\sigma(n_k)}(x) - s, \tag{3.4}$$

hence, letting $k \to +\infty$,

$$b_\sigma \circ \gamma_x (s) = b_\sigma (x) - s. \tag{3.5}$$

Since $b_{\sigma(n_k)}$ is smooth at x, differentiating (3.4) at $s = 0$ gives

$$\left\langle \nabla b_{\sigma(n_k)} (x) , v_{\sigma(n_k)} \right\rangle = -1,$$

and recalling that $|v_{\sigma(n_k)}| = \left|\nabla b_{\sigma(n_k)} (x)\right| = 1$, we obtain[4]

$$v_{\sigma(n_k)} = -\nabla b_{\sigma(n_k)} (x) . \tag{3.6}$$

Similarly[5], using (3.5) we deduce

$$\langle \nabla b_\sigma (x) , v_\sigma \rangle = -1. \tag{3.7}$$

Indeed, fix a normal coordinate chart at x. For the sake of clarity, we use a symbol $\widehat{\ }$ to denote the local expressions in these coordinates so that, e.g., $\widehat{\gamma}_x (s) = sv_\sigma$. By (3.5) we have

$$\lim_{s\to 0} \frac{\widehat{b}_\sigma (sv_\sigma) - \widehat{b}_\sigma (0)}{s} = \frac{d}{ds}\bigg|_{s=0} b_\sigma \circ \gamma_x = -1.$$

On the other hand, since b_σ is differentiable at x and we are working with normal coordinates at x,

$$\lim_{s\to 0} \frac{\widehat{b}_\sigma (sv_\sigma) - \widehat{b}_\sigma (0)}{s} = \nabla \widehat{b}_\sigma (0) \cdot v_\sigma = \langle \nabla b_\sigma (x) , v_\sigma \rangle .$$

We have thus established the validity of (3.7). As above, since $|\nabla b_\sigma (x)| \leq 1$ and $|v_\sigma| = 1$, it follows from (3.7) that

$$|\nabla b_\sigma (x)| = 1$$

and, therefore,

$$v = -\nabla b_\sigma (x) . \tag{3.8}$$

[4] actually, we already know that, for $x \notin \mathrm{cut}(\sigma(t))$, the gradient $\nabla r_{\sigma(t)} (x)$ of the distance function $r_{\sigma(t)} (x) = d(\sigma(t), x)$ at x is just the velocity vector of the minimizing geodesic from $\sigma(t)$ to x.

[5] be careful: if $f \in \mathrm{Lip}(M, \mathbb{R})$ and $u \in \mathrm{W}^{1,p}_{loc}(N, M)$ then $f \circ u \in \mathrm{W}^{1,p}_{loc}$ but, in general, if $\dim M \geq 2$ the chain rule does not hold (even for $u \in C^\infty$). The reason is that u could have big intersections with the non-differentiability set of f, [29]: take, for instance, $M = \mathbb{R}^2$, $f(x, y) = \max(x, y)$, $N = \mathbb{R}$ and $u(t) = (t, t)$. On the other hand, if either $\dim M = 1$ or $\mathrm{rk}(u) = \dim M = \dim N$ then the chain rule holds; see [29] and references therein.

Using the fact that $v_{\sigma(n_k)} \to v$, from (3.6) and (3.8) we finally conclude

$$\nabla b_{\sigma(n_k)}(x) \to \nabla b_{\sigma}(x).$$

This completes the proof. ∎

Example 3.17 *In the standard Euclidean space $\mathbb{R}^m$, consider the ray σ issuing from $o \in \mathbb{R}^m$ along the direction $\xi \in \mathbb{S}^{m-1}$:*

$$\sigma(t) = \xi t + o, \ \ t \geq 0.$$

Then,

$$b_{\sigma(t)}(x) = \frac{|x-o|^2 + 2t\langle o-x, \xi\rangle}{|x-\xi t - o| + t},$$

and letting $t \to +\infty$, we conclude that the corresponding Busemann function takes the form

$$b_{\sigma}(x) = \langle o - x, \xi\rangle.$$

Example 3.18 *Let the standard m-dimensional hyperbolic space $\mathbb{H}^m$ be realized as the Poincarè disc*

$$\mathbb{H}^m = \left(B_1(0), \frac{4\sum dx^i \otimes dx^i}{\left(1-|x|^2\right)^2}\right).$$

The intrinsic distance function satisfies

$$\cosh d_H(x,y) = 1 + \frac{2|x-y|^2}{\left(1-|x|^2\right)\left(1-|y|^2\right)},$$

Consider the ray σ connecting $0 \in B_1(0)$ to the ideal boundary point $\xi \in \partial B_1(0) = \mathbb{S}^{m-1}$:

$$\sigma(t) = \xi \tanh\left(\frac{t}{2}\right), \ \ t \geq 0.$$

We have

$$b_{\sigma(t)}(x) = \operatorname{arccosh}\left\{1 + (\cosh(t)+1)\frac{|x-\xi\tanh(t/2)|^2}{\left(1-|x|^2\right)}\right\} - t.$$

Therefore, letting $t \to +\infty$ we conclude that the corresponding Busemann function takes the form

$$b_{\sigma}(x) = \log\left(\frac{|x-\xi|^2}{1-|x|^2}\right).$$

In the next result, the Ricci curvature assumption enters the game. For the sake of completeness, we first recall the following

Definition 3.19 *Given $u \in W^{1,2}_{loc}(M)$ and $f \in L^1_{loc}(M)$, we say that*

$$\Delta u \le f, \text{ weakly on } M$$

if, for every $0 \le \varphi \in C^\infty_c(M)$, it holds

$$-\int_M \langle \nabla u, \nabla \varphi \rangle \le \int_M f\varphi. \tag{3.9}$$

In case $f = 0$ we say that u is ***weakly superharmonic****.*

Remark 3.20 *For future purposes, it is worth noting that condition (3.9) is equivalent to*

$$\int_M u\Delta\varphi \le \int_M f\varphi,$$

for every $0 \le \varphi \in C^\infty_c(M)$. Indeed, since $u \in W^{1,2}_{loc}(M)$, the divergence theorem applies to the vector field $X = u\nabla\varphi$ and gives

$$-\int_M \langle \nabla\varphi, \nabla u \rangle = \int_M u\Delta\varphi.$$

To see this, take a sequence of smooth functions f_n converging to u in $W^{1,2}(U)$, where U is a compact neighborhood of supp(φ). Then

$$-\int_M \langle \nabla\varphi, \nabla u \rangle = \lim_{n\to+\infty} -\int_M \langle \nabla\varphi, \nabla f_n \rangle = \lim_{n\to+\infty} \int_M \Delta\varphi f_n = \int_M \Delta\varphi u,$$

as claimed.

Lemma 3.21 (Cheeger-Gromoll) *Let $(M, \langle , \rangle)$ be a complete, m-dimensional manifold with $Ric \ge 0$. Let $\sigma : [0, +\infty) \to M$ be a ray. Then, the Busemann function b_σ is weakly superharmonic.*

The proof of Lemma 3.21 is an application of the next result known as the **Laplacian comparison theorem**. For a proof, see the Appendix.

Theorem 3.22 (Laplacian comparison) *Let $(M, \langle , \rangle)$ be a complete manifold of dimension m and let $r(x) = d(x, o)$ be the distance function from a fixed origin $o \in M$. Assume that $Ric \ge (m-1)c$ with $c \in \mathbb{R}$. Set*

$$\mathrm{sn}_c(t) = \begin{cases} \frac{1}{\sqrt{c}} \sin\left(\sqrt{c}t\right) & \text{if } c > 0 \\ t & \text{if } c = 0 \\ \frac{1}{\sqrt{-c}} \sinh\left(\sqrt{-c}t\right) & \text{if } c < 0. \end{cases}$$

Then,

$$(\Delta r)(x) \leq (m-1)\frac{\mathrm{sn}_c'(r(x))}{\mathrm{sn}_c(r(x))}$$

holds pointwise on $M\backslash\mathrm{cut}(o)$ *and weakly on all of* M*.*

Proof (of Lemma 3.21). Recall that $r, b_{\sigma(t)}, b_\sigma \in W^{1,2}_{loc}(M)$, therefore Remark 3.20 applies to each of these functions. Now, let $0 \leq \varphi \in C^\infty_c(M)$. For every fixed $t \geq 0$, by the Laplacian comparion theorem we have

$$\int_M b_{\sigma(t)}\Delta\varphi \leq (m-1)\int_M \frac{\varphi}{d(x,\sigma(t))}.$$

Whence, letting $t \to +\infty$ and using dominated convergence, we deduce

$$\int_M b_\sigma \Delta\varphi = \lim_{t\to+\infty}\int_M b_{\sigma(t)}\Delta\varphi \leq 0.$$

■

It is also convenient to recall the **Bochner formula** for smooth functions; see the Appendix for a proof.

Theorem 3.23 (Bochner formula) *Let* $(M, \langle,\rangle)$ *be a Riemannian manifold and let* $u \in C^\infty(M)$*. Then*

$$\frac{1}{2}\Delta|\nabla u|^2 = |\mathrm{Hess}(u)|^2 + \langle\nabla\Delta u, \nabla u\rangle + Ric(\nabla u, \nabla u).$$

A further basic tool in the proof of the splitting theorem, is the **strong minimum principle** for superharmonic functions.

Theorem 3.24 (strong minimum principle) *Let* $(M, \langle,\rangle)$ *be a Riemannian manifold and let* $u \in C^0(M) \cap W^{1,2}_{loc}(M)$*. If* $\Delta u \leq 0$ *weakly on* M *and* $u(x) \geq u(x_0)$ *for some* $x_0 \in M$*, then* $u(x) \equiv u(x_0)$*.*

We are now ready to give the
Proof (of Theorem 3.13). We show that M is isometric to the Riemannian product $\overline{M} \times \mathbb{R}$ with $\overline{M} \subset M$ a totally geodesic hypersurface. The asserted properties of $\overline{M}$ are obvious.

Let $\sigma : \mathbb{R} \to M$ be a line in M which gives rise to the rays $\sigma^+(t) = \sigma(t) : [0, +\infty) \to M$ and $\sigma^-(t) = \sigma(-t) : [0, +\infty) \to M$. Consider the corresponding Busemann functions b_{σ^+} and b_{σ^-}. Define a new Lipschitz function $u : M \to \mathbb{R}$ by setting

$$b(x) = b_{\sigma^+}(x) + b_{\sigma^-}(x).$$

We claim that $b(x) \geq 0$ on M and that $b(x) = 0$ along σ. Indeed, using the triangle inequality we immediately deduce

(3.10)
$$b_{\sigma^+(t)}(x)+b_{\sigma^-(t)}(x) = d(x,\sigma(t))+d(x,\sigma(-t))-2t \geq d(\sigma(t),\sigma(-t))-2t = 0,$$

for every $t > 0$. Moreover, let $x = \sigma(t_0)$ for some $t_0 \in \mathbb{R}$. Then, for every $t > |t_0|$, we have

(3.11) $$b_{\sigma^+(t)}(x) + b_{\sigma^-(t)}(x) = d(\sigma(t_0),\sigma(t)) + d(\sigma(t_0),\sigma(-t)) - 2t = 0.$$

The claim follows by taking the limits as $t \to +\infty$ in both (3.10) and (3.11). Now, by Theorem 3.21, b is weakly superharmonic. Therefore, by the strong minimum principle we obtain $b(x) \equiv 0$, i.e.

$$b_{\sigma^+}(x) = -b_{\sigma^-}(x).$$

Using once again Theorem 3.21 we conclude that

$$\Delta b_{\sigma^+}(x) = \Delta b_{\sigma^-}(x) = 0.$$

In particular, by elliptic regularity, b_{σ^+} is a smooth harmonic function. Since $|\nabla b_{\sigma^+}| = 1$, inserting b_{σ^+} into the Bochner formula and recalling that $Ric \geq 0$ yields that b_{σ^+} is **affine**, i.e.,

$$\mathrm{Hess}(b_{\sigma^+}) = 0.$$

The announced splitting now follows from the next general result

■

Theorem 3.25 *Let $(M, \langle,\rangle)$ be a complete (connected) Riemannian manifold. Assume that there exists a nonconstant function $u \in C^\infty(M)$ satisfying $\mathrm{Hess}(u) = 0$. Then, M is isometric to the Riemannian product $\overline{M} \times \mathbb{R}$ where $\overline{M}$ is a totally geodesic, connected hypersurface.*

Proof. First of all, note that

$$\left\langle \nabla |\nabla u|^2, X \right\rangle = 2\mathrm{Hess}(u)(\nabla u, X) = 0,$$

for every vector field X. Thus $|\nabla u| = \text{const.}$ and, without loss of generality, we can assume that $|\nabla u| = 1$. Let $\overline{M}$ be any level set of u, say $\overline{M} = \{u = 0\}$. Since u has no critical points, $\overline{M}$ is a smooth hypersurface and $\nu = \nabla u|_{\overline{M}}$ is the Gauss map of $\overline{M}$. In particular, the normal bundle $\nu(\overline{M})$ of $\overline{M}$ in

M is the trivial bundle $\overline{M} \times \mathbb{R}$. Taking the covariant derivative of ν we deduce that the second fundamental form of $\overline{M}$ satisfies $\langle \mathrm{II}_\nu (X,Y), \nu \rangle = \mathrm{Hess}(u)(X,Y) = 0$. Therefore, $\overline{M}$ is totally geodesic. Now, consider the normal exponential map $\exp^\perp : \nu\left(\overline{M}\right) = \overline{M} \times \mathbb{R} \to M$ which is defined by

$$\exp^\perp (x,t) = \exp_x (t\nu(x)).$$

We claim that

$$\exp_x (t\nu(x)) = \phi(t,x), \tag{3.12}$$

where $\phi(t,x)$ denotes the flow of the vector field ∇u. Indeed if $\gamma(t)$ is the integral curve of ∇u issuing from $x \in \overline{M}$ then

$$D_{\dot\gamma}\dot\gamma = D_{\dot\gamma}\nabla u \circ \gamma = \mathrm{Hess}(u)(\dot\gamma,\dot\gamma) = 0.$$

Thus, γ is a geodesic. Since $\gamma(0) = x$ and $\dot\gamma(0) = \nabla u(x) = \nu(x)$ it follows that $\gamma(t) = \exp_x(t\nu(x))$. This proves the validity of (3.12).

Now, it is readily seen that, in general, $\exp^\perp$ is surjective[6]. On the other hand, suppose

$$\exp_{x_1}(t_1\nu(x_1)) = \exp_{x_2}(t_2\nu(x_2)).$$

Since

$$\frac{d}{dt} u \circ \exp_{x_j}(t\nu(x_j)) = \frac{d}{dt} u \circ \phi(t,x_j) = |\nabla u|^2 (\phi(t,x_j)) = 1,$$

and $x_1, x_2 \in \{u = 0\}$, we deduce

$$t_1 = u \circ \exp_{x_j}(t_1\nu(x_1)) = u \circ \exp_{x_j}(t_2\nu(x_2)) = t_2.$$

Let $t = t_1 = t_2$. From the equality

$$\phi(t,x_1) = \phi(t,x_2),$$

using the reverse flow we obtain that $x_1 = x_2$. We have thus shown that $\exp^\perp(x,t)$ is also injective, proving that this map realizes a differential splitting $\overline{M} \times \mathbb{R} \approx M$. Finally, by the Gauss lemma,

$$(\exp^\perp)^* \langle , \rangle = dt \otimes dt + \sigma_{\alpha\beta}(x,t)\, \overline{\theta}^\alpha \otimes \overline{\theta}^\beta, \tag{3.13}$$

[6] this means that a geodesic γ issuing from $x \in M$ and such that $\mathrm{dist}\left(x, \overline{M}\right) = \ell(\gamma)$ must intersect $\overline{M}$ orthogonally. Clearly, γ exists by the geodesic completeness of M.

where $\overline{\langle,\rangle} = \sum_{\alpha=2}^{m} \overline{\theta}^{\alpha} \otimes \overline{\theta}^{\alpha}$ is the induced metric on $\overline{M}$ and the local o.n. coframe $\{\overline{\theta}^{\alpha}\}$ has been parallel translated along the $\mathbb{R}$ component. We observe that, by equality (3.12) and by the very definition of Lie derivative,

$$\frac{d}{dt}(\exp^{\perp})^{*}\langle,\rangle = \frac{d}{dt}\phi_t^{*}\langle,\rangle = L_{\nabla u}\langle,\rangle = \operatorname{Hess}(u) = 0.$$

Using this information into (3.13) we deduce

$$\frac{\partial \sigma_{\alpha\beta}(x,t)}{\partial t} = 0.$$

Since $\sigma_{\alpha\beta}(x,0) = \delta_{\alpha\beta}$, integrating we conclude

$$\sigma_{\alpha\beta}(x,t) = \delta_{\alpha\beta}$$

and (3.13) takes the announced form

$$(\exp^{\perp})^{*}\langle,\rangle = dt^2 + \delta_{\alpha\beta}\overline{\theta}^{\alpha} \otimes \overline{\theta}^{\beta} = dt^2 + \overline{\langle,\rangle}.$$

This completes the proof. ■

We conclude the section by showing how Theorem 3.13 implies the validity of Theorem 3.14. The following preliminary discussion will be useful.

Lemma 3.26 (Cheeger-Gromoll) *Let N be a complete manifold which contains no lines. Then*

$$\mathcal{I}(N \times \mathbb{R}^k) = \mathcal{I}(N) \times \mathcal{I}(\mathbb{R}^k). \tag{3.14}$$

Proof. First of all, observe that given a minimizing geodesic $\sigma(t) = (\sigma_1(t), \sigma_2(t)) : [0,1] \to M_1 \times M_2$ in a generic Riemannian product $M_1 \times M_2$, its components $\sigma_j : [0,1] \to M_j$, $j = 1,2$, are minimizing geodesics. Indeed, by definition of the product structure, σ_1 and σ_2 are both geodesics, hence constant speed curves. In particular,

$$\ell(\sigma) = \int_0^1 \sqrt{|\dot{\sigma}_1(t)|^2 + |\dot{\sigma}_2(t)|^2}dt = \sqrt{\ell(\sigma_1)^2 + \ell(\sigma_2)^2}.$$

The claim now follows easily.

Let us prove the validity of (3.14). Clearly, $\mathcal{I}(N) \times \mathcal{I}(\mathbb{R}^k) \subseteq \mathcal{I}(N \times \mathbb{R}^k)$ therefore we will focus our attention on the opposite implication. Let $f \in \mathcal{I}(N \times \mathbb{R}^k)$. Then f maps lines of $N \times \mathbb{R}^k$ onto lines. Note that, by the assumptions on N, a line σ in $N \times \mathbb{R}^k$ must be of the form $\sigma(t) = (x_0, y(t))$ for some $x_0 \in N$ and for some line $y(t)$ in $\mathbb{R}^k$. It follows that $f \circ \sigma(t) =$

$(x_0', y'(t))$ for some $x_0' \in N$ and some line $y'(t)$ in $\mathbb{R}^k$. This easily implies that f maps (isometrically) $x_0 \times \mathbb{R}^k$ onto $x_0' \times \mathbb{R}^k$ and, since $N \times y_0$ is orthogonal to every $x \times \mathbb{R}^k$, we also deduce that f maps isometrically each slice $N \times y_0$ onto some slice $N \times y_0'$. Now, let $Q_N : N \times \mathbb{R}^k \to N$ and $Q_{\mathbb{R}^k} : N \times \mathbb{R}^k \to \mathbb{R}^k$ denote the standard projections and define $f_N(x,y) = Q_N \circ f(x,y)$ and $f_{\mathbb{R}^k}(x,y) = Q_{\mathbb{R}^k} \circ f(x,y)$. By the previous discussion we have that, for every $x_0 \in N$ and $y_0 \in \mathbb{R}^k$ fixed, it holds $f_N(x, y_0) \in \mathcal{I}(N)$ and $f_{\mathbb{R}^k}(x_0, y) \in \mathcal{I}(\mathbb{R}^k)$. On the other hand, it is easy to show that, in fact, $f_N(x,y)$ is independent of y and $f_{\mathbb{R}^k}(x,y)$ does not depend on x. For instance, let us verify the first claim. To this end, take any curve $y(t)$ in $\mathbb{R}^k$ and compute

$$\frac{d}{dt} f(x_0, y(t)) = \frac{d}{dt} \left(x_0', y'(t)\right) = (0, \frac{d}{dt} y'(t)),$$

for some $x_0' \in N$ and $y'(t) \in \mathbb{R}^k$. On the other hand

$$\frac{d}{dt} f(x, y(t)) = (\frac{d}{dt} f_M(x, y(t)), \frac{d}{dt} f_{\mathbb{R}^k}(x, y(t))).$$

proving that $\frac{d}{dt} f_M(x, y(t)) = 0$, as claimed. In conclusion,

$$f(x,y) = (f_M(x), f_{\mathbb{R}^k}(y)) \in \mathcal{I}(N) \times \mathcal{I}(\mathbb{R}^k),$$

and this completes the proof. ■

Proof (of Theorem 3.14). Since the Riemannian universal covering M' of M is a complete manifold satisfying $Ric' \geq 0$, an iterative application of Theorem 3.13 gives that M' splits isometrically as a product $N \times \mathbb{R}^k$, where N does not contain a line. Therefore, it remains to show that N is necessarily compact. By contradiction, suppose that the complete manifold N is non-compact. Using a construction similar to that presented in Example 3.12, we are going to show that N must contain a line, thus reaching the desired contradiction. Since N is non-compact, as explained in Example 3.10, there exists a ray $\sigma : [0, +\infty) \to N$ issuing from a chosen $o \in N$. Fix any sequence $T^j \nearrow +\infty$ and define a corresponding sequence of minimizing geodesics $\sigma^j(t) = \sigma\left(t + T^j\right) : [-T^j, +\infty) \to N$. Let $D = \operatorname{diam}(M)$, $o' = (o, 0) \in M'$ and recall from (the proof of) Theorem 2.21 that the translates $\{\gamma \overline{B_D'(o')} : \gamma \in \Gamma\}$ cover M'. Here, as usual, $\Gamma = \pi_1(M) \subset \mathcal{I}(M')$ denotes the covering transformation group. Let F be the projection on N of the compact ball $\overline{B_D'(o')}$. Since, by Lemma 3.26, every $\gamma \in \Gamma$ writes as $\gamma = (\gamma_N, \gamma_{\mathbb{R}^k})$ with $\gamma_N \in \mathcal{I}(N)$, it follows that the translates $\{\gamma_N F : \gamma \in \Gamma\}$ must cover N.

Therefore, for each j, we find γ_N^j such that $(\gamma_N^j \sigma^j)(0) \in F$. Set

$$o^j = (\gamma_N^j \sigma^j)(0)$$
$$v^j = \frac{d}{dt}(\gamma_N^j \sigma^j)(0),$$

so that $(\gamma_N^j \sigma^j)(t) = \exp_{o^j}^N (v^j t) : [-T^j, +\infty) \to N$. Clearly, since γ_N^j is an isometry of N, the curve $\gamma_N^j \sigma^j$ is a unit speed minimizing geodesic. From the fact that

$$o^j \in F \text{ and } v^j \in \mathbb{S}_{o^j}^{m-k-1} \subset T_{o^j} N,$$

we deduce that there exists a subsequence

$$\left(o^{j_h}, v^{j_h}\right) \overset{TN}{\to} (o, v).$$

The corresponding sequence of minimizing geodesics $(\gamma_N^{j_h} \sigma^{j_h})(t)$ converges to the line $\sigma(t) = \exp_o^N(tv) : \mathbb{R} \to N$. This contradicts the assumption that N does not contain any line. ∎

Chapter 4

More on discrete group actions: Dirichlet domains

In this section, following the expositions in [41] and [3], we take a closer look at covering transformation groups by focalizing our attention on fundamental domains. Thought the section, until otherwise specified, $\Gamma \subset \mathcal{I}(M)$ will always denote a discrete subgroup of the isometry group of a complete Riemannian manifold $(M, \langle , \rangle)$.

Definition 4.1 *A* ***fundamental domain*** *for the action of* Γ *on* M *is a connected, open set* $D \subset M$ *such that*

(i) $\gamma D \cap D = \emptyset$, for every $1 \neq \gamma \in \Gamma$;

(ii) $\underset{\gamma \in \Gamma}{\cup} \gamma \overline{D} = M$.

Beside fundamental domains, we have generic fundamental sets.

Definition 4.2 *A* ***fundamental set*** *for the action of* Γ *on* M *is a subset* $S \subset M$ *which contains exactly one point of each* Γ*-orbit in* M*.*

In the next lemma we collect some general[1] properties of fundamental sets and domains. Note that properties (I.a) and (I.b) below show that, up to somewhat "negligible" sets, a fundamental domain contains almost all the information about the quotient manifold. Furthermore, according to (II), fundamental sets are useful to verify that some connected open set is actually a fundamental domain.

[1] say topological, i.e., independent of the metric structure of M.

Lemma 4.3 *Let $(M, \langle,\rangle)$ be a complete Riemannian manifold and let $\Gamma \subset \mathcal{I}(M)$ be a discrete group acting on M.*

(I) Let D be a fundamental domain of the action. Then:

(I.a) For every $\gamma \in \Gamma \backslash \{1\}$, $\gamma\overline{D} \cap \overline{D} \subset \partial D$.

(I.b) The inclusion $i : \overline{D} \hookrightarrow M$ induces a continuous bijection $k : \overline{D}/\Gamma \to M/\Gamma$. Furthermore, if the action is proper and $\overline{D}$ is compact, then k is a homeomorphism.

(II) Assume that the action is free. Then the connected, open set $D \subset M$ is a fundamental domain if and only if there exists a fundamental set $S \subset M$ such that

$$D \subseteq S \subseteq \overline{D}.$$

Remark 4.4 *Actually, one can relax the assumption in (I.b) by requiring that the family $\left\{\gamma\overline{D} : \gamma \in \Gamma\right\}$ is locally finite, [41].*

Proof. (I.a) Let $\gamma \neq 1$ and let $x \in \gamma\overline{D} \cap \overline{D}$. We show that $x \notin D$. By contradiction, suppose that $x \in D$. Since D is a neighborhood of $x \in \gamma\overline{D} = \overline{\gamma D}$, we have $D \cap \gamma D \neq \emptyset$. Contradiction.

(I.b) The map k is well defined via the following commutative diagram

$$\begin{array}{ccc} \overline{D} & \overset{i}{\hookrightarrow} & M \\ P|_{\overline{D}} \downarrow & \circlearrowleft & \downarrow P \\ \overline{D}/\Gamma & \overset{k}{\to} & M/\Gamma, \end{array}$$

i.e.

$$k\left(\Gamma x \cap \overline{D}\right) = \Gamma x.$$

Since $\cup_{\gamma\in\Gamma}\gamma\overline{D} = M$, the map k is surjective. On the other hand, $\Gamma x_1 = \Gamma x_2$ forces $\Gamma x_1 \cap \overline{D} = \Gamma x_2 \cap \overline{D}$ and k is also injective. Moreover, by the universal property of the quotient topology, k is continuous if and only if $k \circ P|_{\overline{D}}$ is continuous and this is the case because $k \circ P|_{\overline{D}} = P \circ i$. Finally, assume that $\overline{D}$ is compact. Then both $\overline{D}/\Gamma$ and M/Γ are compact. Moreover, since the action is proper, M/Γ is Hausdorff. A standard compact-Hausdorff argument now shows that the continuous bijection k is a closed, hence an open, map. This proves that k^{-1} is continuous.

(II) Note that, by definition of fundamental set,

$$\cup_{\gamma\in\Gamma}\gamma S = M.$$

Furthermore S contains at most one point of each Γ-orbit of M. Thus, if $x \in S \cap \gamma S$ then $\gamma^{-1}x, x \in S \cap \Gamma x$ which, in turn, implies $\gamma^{-1}x = x$. Since the action is free we deduce that $\gamma = 1$. Therefore

$$S \cap \gamma S = \emptyset, \forall \gamma \neq 1.$$

and the desired properties of D follow from the fact that $D \subseteq S \subseteq \overline{D}$.

Conversely, let D be a fundamental domain. Since $\gamma D \cap D = \emptyset$ for every $\gamma \neq 1$, then D contains at most one point in each Γ-orbit. On the other hand, since $\cup_{\gamma \in \Gamma} \gamma \overline{D} = M$, using the axiom of choice, we can form a fundamental set S by adding to each D a suitable point chosen in $\gamma \partial D$, $\gamma \in \Gamma$. By construction, $D \subseteq S \subseteq \overline{D}$. ■

In case the discrete group Γ acts freely and properly on the complete manifold M, there is a standard procedure to produce a metric(!) fundamental domain for the action. First, we record the following

Definition 4.5 *Fix $x_0 \in M$ and, for every $\gamma \in \Gamma$, let*

$$H_\gamma (x_0) = \{x \in M : d(x, x_0) < d(x, \gamma x_0)\}.$$

Then, the set

$$D(x_0) = \cap_{\gamma \in \Gamma} H_\gamma (x_0)$$

is called a ***Dirichlet domain with center*** *x_0.*

We can imagine $H_\gamma (x_0)$ as an open half-space in M. In this picture, $D(x_0)$ is a kind of polyhedral set.

Example 4.6 *Let $\Gamma = \mathbb{Z}^2$ act on $\mathbb{R}^2$ by translations along the axes. If $x_0 \equiv (1/2, 1/2)$, then $D(x_0)$ is the open square $\left\{(x, y) \in \mathbb{R}^2 : 0 < x < 1,\ 0 < y < 1\right\}$. The orbit space $\overline{D(x_0)}/\Gamma$ is homeomorphic to the 2-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.*

Example 4.7 *Let $\Gamma = \mathbb{Z}_2$ act on $\mathbb{S}^2$ by reflection with respect to $0 \in \mathbb{R}^3$. If $x_0 \equiv (0, 0, 1)$, then $D(x_0)$ is the open hemisphere $\left\{(x, y, z) \in \mathbb{S}^2 : z > 0\right\}$. The orbit space $\overline{D(x_0)}/\Gamma$ is homeomorphic to the real projective plane $\mathbb{RP}^2 = \mathbb{S}^2/\Gamma$.*

Example 4.8 *Let $\Gamma = \mathbb{Z}_4$ act on $\mathbb{R}^2$ by rotations with center $0 \in \mathbb{R}^2$. If $x_0 \equiv (1/2, 1/2)$, then $D(x_0)$ is the open quadrant $\left\{(x, y) \in \mathbb{R}^2 : y > x\right\}$. The orbit space $\overline{D(x_0)}/\Gamma$ is homeomorphic to the cone*

$$\mathbb{R}^2/\Gamma = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2,\ z \geq 0\right\}.$$

The following Lemma will be useful to investigate the basic properties of Dirichlet domains.

Lemma 4.9 *Let $(M, \langle , \rangle)$ be a complete manifold. Keeping the above notation, we have:*

(a) Every (minimizing) geodesic segment $\sigma : [0, L] \to M$ issuing from $\sigma(0) = x_0$ cannot intersect $\partial H_\gamma(x_0)$ twice. Furthermore, if σ intersects $\partial H_\gamma(x_0)$ at some $\sigma(t_0)$, $0 < t_0 < L$, then the intersection is "non-tangential" in the sense that $\sigma(t) \in M \backslash \overline{H_\gamma(x_0)}$ for $t_0 < t \leq L$.

(b) The open set $H_\gamma(x_0)$ is star-shaped with respect to x_0, i.e., every minimizing geodesic connecting x_0 with a point $x \in H_\gamma(x_0)$ must be contained in $H_\gamma(x_0)$.

(c) The boundary set $\partial H_\gamma(x_0)$ has (Riemannian) measure zero.

Proof. (a) By contradiction, suppose that $\sigma(t_1), \sigma(t_2) \in \partial H_\gamma(x_0)$ for some $t_1 < t_2$. Necessarily, $\sigma(t_1) \neq \sigma(t_2)$. Then, using the triangle inequality and the definition of $H_\gamma(x_0)$ we obtain

$$\begin{aligned} d(\gamma x_0, \sigma(t_2)) &= d(x_0, \sigma(t_2)) \\ &= d(x_0, \sigma(t_1)) + d(\sigma(t_1), \sigma(t_2)) \\ &= d(\gamma x_0, \sigma(t_1)) + d(\sigma(t_1), \sigma(t_2)), \end{aligned}$$

proving that $\gamma x_0, \sigma(t_1)$ and $\sigma(t_2)$ are collinear distinct points with $\sigma(t_1)$ between γx_0 and $\sigma(t_1)$. Since also $x_0, \sigma(t_1)$ and $\sigma(t_2)$ are collinear distinct points with $\sigma(t_1)$ between x_0 and $\sigma(t_1)$, recalling that geodesics cannot bifurcate, we conclude $x_0 = \gamma x_0$. The contradiction proves that σ can intersect $\partial H_\gamma(x_0)$ only once. Suppose now that $\sigma(t_0) \in \partial H_\gamma(x_0)$ and $\sigma(t) \in H_\gamma(x_0)$ for every $t \neq t_0$. Then,

$$\begin{aligned} d(x_0, \sigma(L)) &= d(x_0, \sigma(t_0)) + d(\sigma(t_0), \sigma(L)) \\ &= d(\gamma x_0, \sigma(t_0)) + d(\sigma(t_0), \sigma(L)) \\ &\geq d(\gamma x_0, \sigma(L)). \end{aligned}$$

On the other hand, since $\sigma(L) \in H_\gamma(x_0)$ we have

$$d(x_0, \sigma(L)) < d(\gamma x_0, \sigma(L)),$$

contradiction. We have thus proved that $\sigma(t) \in (M \backslash H_\gamma(x_0))$ for every $t > t_0$.

(b) Straightforward consequence of (a).

(c) Since the closed set $\mathrm{cut}\,(x_0) \cup \mathrm{cut}\,(\gamma x_0)$ has measure zero, it suffices to show that $N = \partial H_\gamma\,(x_0) \cap (M \backslash \mathrm{cut}\,(x_0)) \cap (M \backslash \mathrm{cut}\,(\gamma x_0))$ has measure zero. Consider the smooth function $f = r_{x_0} - r_{\gamma x_0} : M \backslash (\mathrm{cut}\,(x_0) \cup \mathrm{cut}\,(\gamma x_0)) \to \mathbb{R}$ where r_{x_0} and $r_{\gamma x_0}$ denote the distance functions from x_0 and γx_0, respectively. Since $N = \{f = 0\}$ the desired conclusion follows once we show that N is a smooth embedded hypersurface of $M \backslash (\mathrm{cut}\,(x_0) \cup \mathrm{cut}\,(\gamma x_0))$. By the implicit function theorem, we are reduced to prove that f has no critical points in N. Let $x \in N$. Since both r_{x_0} and $r_{\gamma x_0}$ are smooth at x, and $d\,(x, x_0) = L = d\,(x, \gamma x_0)$, we have $\nabla r_{x_0}\,(x) = \dot{\sigma}\,(L)$ and $\nabla r_{\gamma x_0}\,(x) = \dot{\delta}\,(L)$ where $\sigma, \delta : [0, L] \to M$ are the(!) unit speed, minimizing geodesics connecting x_0 to x and γx_0 to x, respectively. Note also that, by the choice of x, both these geodesics are minimizing past L, say on a bigger interval $[0, L + \varepsilon]$. Now, according to (a), x is the unique intersection point of σ and δ with $\partial H_\gamma\,(x_0)$. Moreover, for $L < t \leq L + \varepsilon$, $\sigma\,(t) \in (M \backslash \overline{H_\gamma\,(x_0)})$ while $\delta\,(t) \in H_\gamma\,(x_0)$. It follows that $\dot{\sigma}\,(L) \neq \dot{\delta}\,(L)$, i.e., $\nabla f\,(x) = \nabla r_{x_0}\,(x) - \nabla r_{\gamma x_0}\,(x) \neq 0$ as claimed. ■

We are now in the position to prove the main result of the section.

Theorem 4.10 *Assume that Γ acts freely and properly by isometries on the complete Riemannian manifold $(M, \langle,\rangle)$, hence $P : M \to N = M/\Gamma$ is a Riemannian covering. Let $x_0 \in M$ be any fixed point. Then:*

(a) The Dirichlet domain $D\,(x_0)$ is a fundamental domain for the action and the family $\left\{\gamma \overline{D\,(x_0)} : \gamma \in \Gamma\right\}$ is locally finite. In particular, $\overline{D\,(x_0)}/\Gamma$ is homeomorphic to M/Γ.

(b) The set $\partial D\,(x_0)$ has measure zero. In particular, for every $\gamma \in \Gamma \backslash \{1\}$, the set $\gamma \overline{D\,(x_0)} \cap \overline{D\,(x_0)}$ has measure zero.

(c) For every $R > 0$,

$$P(B_R^M\,(x_0) \cap \overline{D\,(x_0)}) = B_R^N\,(P\,(x_0))$$

and

$$\mathrm{Vol}\left(B_R^M\,(x_0) \cap \overline{D\,(x_0)}\right) = \mathrm{Vol}\left(B_R^N\,(P\,(x_0))\right).$$

Proof. (a) Let us divide the verification in steps.

(a.1) $D\,(x_0)$ is open. We prove that $M \backslash D\,(x_0) = \cup_{\gamma \in \Gamma}\,(M \backslash H_\gamma\,(x_0))$ is closed. This follows once we show that the family of closed sets $\{M \backslash H_\gamma\,(x_0)\}_{\gamma \in \Gamma}$

is locally finite. Fix any $x \in M$. We prove that

$$(M \backslash H_\gamma (x_0)) \cap B_1 (x) \subseteq \gamma \overline{B_R (x)}, \tag{4.1}$$

where $R = 1 + 2d(x, x_0)$. This obviously imply

$$(M \backslash H_\gamma (x_0)) \cap B_1 (x) \subseteq \gamma \overline{B_R (x)} \cap \overline{B_R (x)}$$

and, since the action is proper and $\overline{B_R (x)}$ is compact, we conclude that $B_1 (x)$ can intersect at most a finite number of sets $M \backslash H_\gamma (x_0)$, as desired. Now, for the proof of (4.1), let $y \in (M \backslash H_\gamma (x_0)) \cap B_1 (x) \neq \emptyset$. Then, using the triangle inequality and the definition of $H_\gamma (x_0)$, we obtain

$$d(y, \gamma x) \leq d(y, x) + d(x, \gamma x) \leq 1 + d(x, \gamma x_0) + d(\gamma x_0, \gamma x) \leq 1 + 2d(x, x_0) = R,$$

showing that $y \in \gamma \overline{B_R (x)}$.

(a.2) $D(x_0)$ is connected[2]. By (b) of Lemma 4.9, each set $H_\gamma (x_0)$ is star-shaped with respect to x_0. Therefore $D(x_0) = \cap_{\gamma \in \Gamma} H_\gamma (x_0)$ is star shaped with respect to x_0.

(a.3) $D(x_0)$ is a fundamental domain. According to (II) of Lemma 4.3, we have to prove that there exists a fundamental set S satisfying

$$D(x_0) \subseteq S \subseteq \overline{D(x_0)}.$$

To this end, for every $x \in M$, choose your favorite $\gamma_x \in \Gamma$ such that

$$\operatorname{dist}(x_0, \Gamma x) = d(x_0, \gamma_x x),$$

or, equivalently,

$$\operatorname{dist}(\Gamma x_0, x) = d\left(\gamma_x^{-1} x_0, x\right).$$

Then, the fundamental set[3]

$$S = \{\gamma_x x : \Gamma x \in M/\Gamma\}$$

has the desired property. Indeed, let $x \in D(x_0)$ so that, by definition, $d(x, x_0) < d(x, \gamma x_0)$, for every $\gamma \neq 1$. Then, x is the unique(!) nearest point to Γx_0 forcing $\gamma_x x = 1x = x \in S$. We have thus shown that $D(x_0) \subseteq S$. Conversely, let $\gamma_x x \in S$. Then, by definition of γ_x, we have

$$d(\gamma_x x, x_0) = \operatorname{dist}(\Gamma x, x_0) \leq d\left(\gamma^{-1} \gamma_x x, x_0\right) = d(\gamma_x x, \gamma x_0),$$

[2] warning: $D(x_0)$ is the intersection, not the union, of connected sets containing x_0, therefore it is not obvious that $D(x_0)$ is connected.

[3] by construction, S contains exactly one point of each Γ-orbit.

for every $\gamma \in \Gamma$. It follows that $\gamma_x x \in \overline{H_\gamma(x_0)}$ for every $\gamma \in \Gamma$, which implies $\gamma_x x \in \overline{D(x_0)}$.

(a.4) The family $\left\{\gamma\overline{D(x_0)} : \gamma \in \Gamma\right\}$ is locally finite. To this end, note that if $y \in \gamma\overline{D(x_0)}$ then $\gamma^{-1}y \in \overline{D(x_0)}$ so that, by definition, $d\left(\gamma^{-1}y, x_0\right) \leq d\left(\gamma^{-1}y, \gamma' x_0\right)$ for every $\gamma' \in \Gamma$. In particular, choosing $\gamma' = \gamma^{-1}$ we get

$$y \in \gamma\overline{D(x_0)} \Rightarrow d\left(\gamma^{-1}y, x_0\right) \leq d(y, x_0). \tag{4.2}$$

Now, pick a point $x \in M$. We show that the ball $B_1(x)$ can intersect at most a finite number of the $\gamma\overline{D(x_0)}$'s. Indeed, assume that $y \in \gamma\overline{D(x_0)} \cap B_1(x) \neq \emptyset$. Using (4.2) and the triangle inequality we obtain

$$\begin{aligned} d\left(x_0, \gamma^{-1}x_0\right) &\leq d\left(x_0, \gamma^{-1}y\right) + d\left(\gamma^{-1}y, \gamma^{-1}x_0\right) \\ &\leq 2d(x_0, y) \\ &\leq 2d(x_0, x) + 2d(x, y) \\ &= 2d(x_0, x) + 2. \end{aligned}$$

Therefore, if we set $R = 2d(x_0, x) + 2$, we deduce that

$$\gamma\overline{D(x_0)} \cap B_1(x) \neq \emptyset \Longrightarrow x_0 \in \overline{B_R(x_0)} \cap \gamma^{-1}\overline{B_R(x_0)} \neq \emptyset.$$

The desired conclusion now follows by recalling that the action is proper.

(b) Note that, by definition of $D(x_0)$,

$$\partial D(x_0) \subseteq \cup_{\gamma\in\Gamma} \partial H_\gamma(x_0).$$

Since Γ is countable and, by (c) of Lemma 4.9, each set $\partial H_\gamma(x_0)$ has measure zero, we deduce that $\partial D(x_0)$ has measure zero. To conclude, recall from (a) of Theorem 4.3, that

$$\gamma\overline{D(x_0)} \cap \overline{D(x_0)} \subseteq \partial D(x_0).$$

(c) Since the Riemannian covering map $P : M \to N = M/\Gamma$ is distance decreasing, we have that

$$P\left(B_R^M(x_0) \cap \overline{D(x_0)}\right) \subseteq B_R^N(P(x_0)).$$

On the other hand, let $y \in B_R^N(P(x_0))$ and let $\sigma^N : [0,1] \to N$ be a minimizing geodesic, of length $\ell(\sigma) < R$, connecting $\sigma^N(0) = P(x_0)$ with $\sigma^N(1) = y$. Let $\sigma^M : [0,1] \to M$ be the unique lifting of σ^N issuing from

x_0. Since P is a local isometry, $\ell\left(\sigma^M\right) = \ell\left(\sigma^N\right) < R$ proving that $\sigma^M(1) = x \in B_R^M(x_0) \cap P^{-1}(y)$. Since $D(x_0)$ is a fundamental domain, there exists $\gamma \in \Gamma$ such that $x \in \gamma\overline{D(x_0)}$. Set $x = \gamma x'$ with $x' \in \overline{D(x_0)}$. Then, using (4.2), we conclude

$$d\left(x', x_0\right) = d\left(\gamma^{-1}x, x_0\right) \leq d\left(\gamma^{-1}x, \gamma^{-1}x_0\right) = d(x, x_0) < R$$

proving that $x' \in B_R^M(x_0) \cap \overline{D(x_0)}$. Since $P(x') = P(x) = y$ we have shown that

$$B_R^N(P(x_0)) \subseteq P\left(B_R^M(x_0) \cap \overline{D(x_0)}\right).$$

Therefore,

$$P\left(B_R^M(x_0) \cap \overline{D(x_0)}\right) = B_R^N(P(x_0)),$$

as desired. Now, observe that P is an isometry of $D(x_0)$ onto its image. Moreover, according to (b), $\mathrm{Vol}(\partial D(x_0)) = 0$ which, by Sard theorem, implies $\mathrm{Vol}(P(\partial D(x_0))) = 0$. It follows that

$$\begin{aligned}
\mathrm{Vol}(B_R^M(x_0) \cap \overline{D(x_0)}) &= \mathrm{Vol}\left(B_R^M(x_0) \cap D(x_0)\right) \\
&= \mathrm{Vol}\left(P\left(B_R^M(x_0) \cap D(x_0)\right)\right) \\
&= \mathrm{Vol}(P(B_R^M(x_0) \cap \overline{D(x_0)})) \\
&= \mathrm{Vol}\left(B_R^N(P(x_0))\right),
\end{aligned}$$

and this completes the proof. ■

By way of example, in conclusion of this section, let us apply Theorem 4.10 to deduce the area formula for finite coverings; see Theorem 1.8.

Proposition 4.11 *Let $(M, \langle,\rangle)$ be a compact Riemannian manifold and let $\Gamma \subset \mathcal{I}(M)$ be a finite group of isometries acting freely on M. Endow the compact manifold $N = M/\Gamma$ with the induced Riemannian metric. Then*

$$\mathrm{Vol}(M) = |\Gamma| \, \mathrm{Vol}(N).$$

Proof. Introduce the Riemannian covering map $P : M \to N = M/\Gamma$. Fix $x_0 \in M$ and let $D(x_0)$ be the corresponding Dirichlet domain with center at x_0. Since $D(x_0)$ is a fundamental domain, by (I.b) of Lemma 4.3 , we have $N = \overline{D(x_0)}/\Gamma$. Thus,

$$N = P(\overline{D(x_0)}) = P(D(x_0)) \cup P(\partial D(x_0)).$$

Now, P is an isometry on $D(x_0)$, therefore

$$\mathrm{Vol}(D(x_0)) = \mathrm{Vol}(P(D(x_0))).$$

Moreover, we know from (b) of Theorem 4.10 that $\mathrm{Vol}(\partial D(x_0)) = 0$ which implies, by Sard theorem, $\mathrm{Vol}(P(\partial D(x_0))) = 0$. We have thus shown that

$$\mathrm{Vol}(N) = \mathrm{Vol}(D(x_0)) = \mathrm{Vol}\left(\overline{D(x_0)}\right).$$

Whence, recalling also that, for every $\gamma \neq 1$,

$$\mathrm{Vol}\left(\gamma\overline{D(x_0)} \cap \overline{D(x_0)}\right) = 0,$$

we conclude

$$|\Gamma| \,\mathrm{Vol}(N) = \sum_{\gamma\in\Gamma} \mathrm{Vol}\left(\gamma\overline{D(x_0)}\right) = \mathrm{Vol}\left(\underset{\gamma\in\Gamma}{\cup}\gamma\overline{D(x_0)}\right) = \mathrm{Vol}(M).$$

■

Chapter 5

Fundamental group of a complete manifold with $Ric \geq 0$

The main result of the section is the following striking quantitative estimate by M. Anderson, [3] and P. Li, [28]. Compare with the compact case considered in Theorem 1.7 above. The proof is from Anderson paper. The arguments used by Li rely on heat kernel methods.

Theorem 5.1 (Anderson, Li) *Let $(M, \langle , \rangle)$ be a complete, non-compact, Riemannian manifold of dimension m. Assume that $Ric \geq 0$ and $\mathrm{Vol}\mathbf{B}_r(0) \preccurlyeq \mathrm{Vol}B_r(o)$, where $\mathbf{B}_r(0)$ denotes the ball of radius r in the Euclidean space $\mathbb{R}^m$. Then,*

$$|\pi_1(M)| \leq \lim_{R \to +\infty} \frac{\mathrm{Vol}\mathbf{B}_R(0)}{\mathrm{Vol}B_R(o)} < +\infty. \tag{5.1}$$

In particular, if $2 > \lim_{R\to+\infty} \mathrm{Vol}\mathbf{B}_R(0) / \mathrm{Vol}B_R(o)$, then M is simply connected. Moreover, the equality holds in (5.1) if and only if M is isometric to $\mathbb{R}^m$.

Remark 5.2 *Since M is assumed to satisfy $Ric \geq 0$, by the Bishop-Gromov comparison theorem it holds $\mathrm{Vol}B_r(o) \leq \mathrm{Vol}\mathbf{B}_r(0)$. Therefore, assumption $\mathrm{Vol}\mathbf{B}_r(0) \preccurlyeq \mathrm{Vol}B_r(o)$ implies that M has* ***maximal volume growth.***

The proof consists of two parts. First, by strengthening Švarc-Milnor arguments, it is shown that every finitely generated subgroup of $\pi_1(M)$ is finite. Next, it is proved that there exits a uniform bound for the cardinality of the finitely generated, hence finite, subgroups of $\pi_1(M)$. This implies that $|\pi_1(M)|$ itself satisfies the same bound and, in particular, is finite.

Example 5.3 *Following Anderson, consider the Abelian group $\mathbb{Q}/\mathbb{Z}$ and note that it is a torsion group, i.e., all its elements have finite period. Indeed $n\mathbb{Z}\frac{m}{n}$ is the unit in $\mathbb{Q}/\mathbb{Z}$. In particular, every finitely generated subgroup of $\mathbb{Q}/\mathbb{Z}$ is finite. However $\mathbb{Q}/\mathbb{Z}$ is infinite. Clearly, there is no uniform bound on the cardinality of the finite subgroups of $\mathbb{Q}/\mathbb{Z}$.*

According to the above strategy we begin by proving the following extended version of Theorem 3.1. Recall that, having fixed an origin $x_0 \in M$, $v_M(r) = \mathrm{Vol}B_r(x_0)$ is the growth function of M. Obviously, the growth type of $v_M(r)$ does not depend on the chosen point x_0. Furthermore, recall that for a finitely generated group Γ, the corresponding growth function is denoted by $\beta_\Gamma(r)$. Its growth type does not depend on the chosen finite set of generators.

Theorem 5.4 (Anderson) *Let $(M, \langle,\rangle)$ be a complete Riemannian manifold of dimension m and let $\Gamma \subset \mathcal{I}(M)$ be a finitely generated, discrete group acting freely and properly on M. The corresponding Riemannian covering map is denoted by $P : M \to N = M/\Gamma$. Assume that $v_M(r) \preccurlyeq r^{k+h}$ and $r^h \preccurlyeq v_N(r)$. Then $\beta_\Gamma(r) \preccurlyeq r^k$.*

Remark 5.5 *In case M is the universal covering of N and $\Gamma \subset \pi_1(N)$ is a finitely generated subgroup, if we assume that $Ric \geq 0$ then by Bishop-Gromov we have $v_M(r) \preccurlyeq r^m$. Applying Theorem 5.4 with $h = 0$ shows that Γ has polynomial growth of degree $\leq m$. We have thus recovered Svarc-Milnor original result.*

Proof. We follow closely the proof of Theorem 3.1. The only difference relies in the clever use of Dirichlet domains and the volume property (c) of Theorem 4.10. Having fixed an origin $x_0 \in M$ and a finite set of generators $S = S^{-1}$ of Γ, let

$$\mu = \max\{d_M(x_0, sx_0) : s \in S\}$$

and recall that, for every $\gamma \in \overline{\mathcal{B}_r(1)} \subset (\Gamma, d_S)$,

$$d_M(x_0, \gamma x_0) \leq \mu r.$$

Let $D(x_0)$ be the Dirichlet domain for the action of Γ. It follows that

$$\begin{aligned}\left|\overline{\mathcal{B}_r(1)}\right| \operatorname{Vol}\left(B_r^M(x_0) \cap \overline{D(x_0)}\right) &= \sum_{\gamma \in \mathcal{B}_r(1)} \operatorname{Vol}\left(B_r^M(\gamma x_0) \cap \gamma \overline{D(x_0)}\right) \\ &\leq \sum_{\gamma \in \mathcal{B}_r(1)} \operatorname{Vol}\left(B_{\mu r + r}^M(x_0) \cap \gamma \overline{D(x_0)}\right) \\ &\leq \sum_{\gamma \in \Gamma} \operatorname{Vol}\left(B_{\mu r + r}^M(x_0) \cap \gamma \overline{D(x_0)}\right) \\ &= \operatorname{Vol}\left(\cup_{\gamma \in \Gamma}\left(B_{\mu r + r}^M(x_0) \cap \gamma \overline{D(x_0)}\right)\right) \\ &= \operatorname{Vol} B_{\mu r + r}^M(x_0).\end{aligned}$$

On the other hand,

$$\operatorname{Vol}\left(B_r^M(x_0) \cap \overline{D(x_0)}\right) = \operatorname{Vol} B_r^N(P(x_0)).$$

Therefore

$$\left|\overline{\mathcal{B}_r(1)}\right| \operatorname{Vol} B_r^N(P(x_0)) \leq \operatorname{Vol} B_{\mu r + r}^M(x_0)$$

that is

$$\left|\overline{\mathcal{B}_r(1)}\right| \leq \frac{\operatorname{Vol} B_{\mu r + r}^M(x_0)}{\operatorname{Vol} B_r^N(P(x_0))}.$$

The desired conclusion now follows from the volume growth assumptions on M and N. ■

We are now in the position to complete the proof of Theorem 5.1.

Proof (of Theorem 5.1). Introduce the Riemannian universal covering $P : M' \to M = M'/\Gamma$ where, as usual, $\Gamma = \pi_1(M) \subset \mathcal{I}(M')$. Let $\Gamma_1 \subset \Gamma$ be any finitely generated subgroup. Note that, by Theorem 5.4, $|\Gamma_1| < +\infty$. Consider the finite-sheeted Riemannian covering $P_1 : M' \to M_1 = M'/\Gamma_1$. Let $o = P(o')$ and $o_1 = P_1(o')$ be fixed. We shall show that

$$|\Gamma_1| \leq \lim_{R \to +\infty} \frac{\operatorname{Vol} \mathbf{B}_R(0)}{\operatorname{Vol} B_R^M(o)} < +\infty. \tag{5.2}$$

To this end, note that

$$R^m \preccurlyeq \operatorname{Vol} B_R^M(o) \leq \operatorname{Vol} B_R^{M_1}(o_1). \tag{5.3}$$

Indeed, the Dirichlet domains $D(o')$ with respect to Γ and $D_1(o')$ with respect to $\Gamma_1 \subset \Gamma$ obviously satisfy

$$D(o') \subseteq D_1(o').$$

Therefore, using (c) of Theorem 4.10,

$$\mathrm{Vol}B_R^M(o) = \mathrm{Vol}(B_R^{M'}(o') \cap \overline{D\,(o')}) \leq \mathrm{Vol}(B_R^{M'}(o') \cap \overline{D_1\,(o')}) = \mathrm{Vol}B_R^{M_1}(o_1)\,,$$

as claimed. Note also that, since Γ_1 is finite, there exists $\mu > 0$ such that, for every $\gamma \in \Gamma_1$,

$$\gamma B_R^{M'}(o') \subseteq B_{R+\mu}^{M'}(o')\,.$$

It follows that

$$\begin{aligned}|\Gamma_1|\,\mathrm{Vol}B_R^{M'}(o') &= \sum_{\gamma\in\Gamma_1} \mathrm{Vol}\left(\gamma B_R^{M'}(o')\right)\\ &= \sum_{\gamma\in\Gamma_1} \mathrm{Vol}\left(\gamma B_R^{M'}(o') \cap \gamma\overline{D_1\,(o')}\right)\\ &\leq \sum_{\gamma\in\Gamma_1} \mathrm{Vol}\left(B_{R+\mu}^{M'}(o') \cap \gamma\overline{D_1\,(o')}\right)\\ &= \mathrm{Vol}B_{R+\mu}^{M'}(o')\,.\end{aligned}$$

On the other hand

$$\mathrm{Vol}B_R^{M'}(o') \geq \mathrm{Vol}\left(B_R^{M'}(o') \cap \overline{D_1\,(o')}\right) = \mathrm{Vol}B_R^{M_1}(o_1)\,.$$

Summarizing, we have obtained

$$|\Gamma_1|\,\mathrm{Vol}B_R^{M_1}(o_1) \leq \mathrm{Vol}B_{R+\mu}^{M'}(o')\,.$$

Now recall that $Ric' \geq 0$ and, therefore, by Bishop-Gromov comparison, $\mathrm{Vol}B_R^{M'}(o') \leq \mathrm{Vol}\mathbf{B}_R(0)$. Moreover, according to (5.3), $\mathrm{Vol}B_R^M(o) \leq \mathrm{Vol}B_R^{M_1}(o_1)$. Therefore, from the above, we deduce

$$\begin{aligned}|\Gamma_1| &\leq \frac{\mathrm{Vol}B_{R+\mu}^{M'}(o')}{\mathrm{Vol}B_R^{M_1}(o_1)}\\ &= \frac{\mathrm{Vol}(B_{R+\mu}^{M'}(o') \setminus B_R^{M'}(o'))}{\mathrm{Vol}B_R^{M_1}(o_1)} + \frac{\mathrm{Vol}B_R^{M'}(o')}{\mathrm{Vol}B_R^{M_1}(o_1)}\\ &\leq \frac{\mathrm{Vol}(B_{R+\mu}^{M'}(o') \setminus B_R^{M'}(o'))}{\mathrm{Vol}B_R^{M_1}(o_1)} + \frac{\mathrm{Vol}\mathbf{B}_R(0)}{\mathrm{Vol}B_R^M(o)}.\end{aligned} \tag{5.4}$$

Since $R^m \preccurlyeq \mathrm{Vol}B_R^{M_1}\,(o_1)$, using also the Bishop-Gromov comparison for the area functions we get (see Remark A.6 in the Appendix)

$$\lim_{R\to+\infty} \frac{\mathrm{Vol}(B_{R+\mu}^{M'}\,(o') \setminus B_R^{M'}\,(o'))}{\mathrm{Vol}B_R^{M_1}\,(o_1)} = 0,$$

and taking the limits in (5.4) as $R \to +\infty$ we conclude

$$|\Gamma_1| \leq \lim_{R\to+\infty} \frac{\mathrm{Vol}\mathbf{B}_R(0)}{\mathrm{Vol}B_R^M(o)}, \tag{5.5}$$

proving (5.2). Since (5.5) holds for every finitely generated (hence finite) subgroup $\Gamma_1 \subset \Gamma$, it follows that Γ is a finite group and the same inequalities are satisfied by Γ itself. This completes the first part of the proof.

Suppose now that

$$|\Gamma| = \lim_{R\to+\infty} \frac{\mathrm{Vol}\mathbf{B}_R(0)}{\mathrm{Vol}B_R^M(o)}.$$

According to (5.4) with $\Gamma_1 = \Gamma$ we have

$$|\Gamma| = \lim_{R\to+\infty} \frac{\mathrm{Vol}\mathbf{B}_R(0)}{\mathrm{Vol}B_R^M(o)} = \lim_{R\to+\infty} \frac{\mathrm{Vol}B_R^{M'}(o')}{\mathrm{Vol}B_R^M(o)}.$$

Therefore

$$\lim_{R\to+\infty} \frac{\mathrm{Vol}\mathbf{B}_R(0)}{\mathrm{Vol}B_R^{M'}(o')} = \lim_{R\to+\infty} \frac{\mathrm{Vol}\mathbf{B}_R(0)}{\mathrm{Vol}B_R^M(o)} \cdot \frac{\mathrm{Vol}B_R^M(o)}{\mathrm{Vol}B_R^{M'}(o')} = 1.$$

Since, by Bishop-Gromov, $R \longmapsto \mathrm{Vol}\mathbf{B}_R(0)\,/\mathrm{Vol}B_R^{M'}(o')$ is increasing and ≥ 1 we deduce

$$\mathrm{Vol}\mathbf{B}_R(0) = \mathrm{Vol}B_R^{M'}\left(o'\right), \, \forall R > 0,$$

and the equality case in the Bishop-Gromov comparison yields $M' = \mathbb{R}^m$. In particular Γ is a finite subgroup of the group of rigid motions $\mathcal{I}(\mathbb{R}^m)$. It is now a standard fact that every element $\gamma \in \Gamma$ must have a fixed point, see Lemma 5.6 below. Since we assume that Γ acts freely on $\mathbb{R}^m$ it follows that $\Gamma = 1$, i.e., $M = M' = \mathbb{R}^m$. ■

Recall from [24] that a **center of mass** of a compact, isometrically immersed submanifold $f : S \to M$ in a complete manifold $(M, \langle,\rangle)$ is a minimum point of the functional

$$I_S(x) = \frac{1}{2\mathrm{Vol}_S(S)} \int_S d(x, f(s))^2 \, d\mathrm{Vol}_S(s) : M \to \mathbb{R}.$$

Clearly, in case S is degenerate, i.e. S is a finite set, $d\mathrm{Vol}_S$ stands for the counting measure. It can be shown that, if S has sufficiently small diameter with respect to the positive part of the curvature of M[1], then the center of mass of S is unique, [24]. For instance, this holds for every(!) compact submanifold in a Cartan-Hadamard manifold.

[1]precisely, this means that $f(S) \subset B_R^M(o)$ where $\max\left\{Sec^M(x) : x \in \overline{B_R^M(o)}\right\} < \pi/2R$.

Lemma 5.6 *Let $(M, \langle , \rangle)$ be a complete manifold with the property that, for every finite set $S \subset M$, the center of mass is unique. If $\Gamma \subset \mathcal{I}(M)$ is a finite group, then every element $\gamma \in \Gamma$ has a fixed point.*

Proof. Let $\gamma \in \Gamma$. Since Γ is finite, $\gamma^k = 1$ for some $k \in \mathbb{N}$. Fix any $x \in M$ and consider the set $S = \left\{x, \gamma x, ..., \gamma^{k-1} x\right\}$. Let $x_0 \in M$ be the center of mass of S. Since $\gamma S = S$ and, by assumption, the center of mass of S is unique we must conclude that $\gamma x_0 = x_0$. ■

Remark 5.7 *A similar proof can be used to show that compact Lie groups G of isometries of a Cartan-Hadamard manifold M have fixed points provided* $\dim G < \dim M$. *Indeed, note that the orbit Gx of a selected point $x \in M$ is realized as an immersed submanifold $\theta_x : G \to M$ such that $\theta_x(g) = gx$.*

Chapter 6

Remarks on the Anderson-Li theorem

In this final Chapter we discuss the role of the Ricci curvature assumption and we give some applications and extensions of Theorem 5.1. Unlike the other parts of these notes, in this Chapter proofs are only sketched or even omitted. We shall try to be quite informative, by compressing in few pages a certain number of tools and works (many of them very deep) taken from somewhat different areas.

6.1 Minimal submanifolds and LCF manifolds

As it was observed by Anderson, in the first part of the proof of Theorem 5.1, the Ricci curvature assumption is merely used to ensure that the covering manifold has polynomial volume growth of degree at most m; see Theorem 5.4. Accordingly, there are other geometric situations where the first conclusion of Theorem 5.1 applies. For instance, this happens for **minimal submanifolds** in the Euclidean space, see [12] and e.g. [37]. Let M be a complete, m-dimensional manifold. Say that **the isometric immersion** $f : M \to \mathbb{R}^n$ **is minimal** if each coordinate function $f^A : M \to \mathbb{R}$ is harmonic. Clearly any Riemannian covering of a minimal submanifold is itself a minimal submanifold. The classical **monotonicity formula** states that, for any fixed origin $o \in M$,

$$r \longmapsto \frac{\mathrm{Vol} B_r(o)}{\mathrm{Vol} \mathbf{B}_r(o)} \quad \text{is increasing.}$$

Here, as usual, $\mathbf{B}_r(o)$ denotes the ball in $\mathbb{R}^m$. In particular, $r^m \preccurlyeq \mathrm{Vol}B_r(o)$. The m-dimensional complete manifold M is said to be of **finite total scalar curvature** if

$$\int_M |Scal|^{\frac{m}{2}} < +\infty.$$

It is known from some work by P. Hartman and K. Shiohama in dimension $m = 2$, [20], [42], and by Anderson in every dimension, [2], that a complete minimal submanifold of finite total scalar curvature satisfies $\mathrm{Vol}B_r(o) \preccurlyeq r^m$. Furthermore, it is also known that M is conformally diffeomorphic to a compact Riemannian manifold N with boundary $\partial N \neq \emptyset$. In particular, M has a finite number of (cylindrical) ends. Using the proof of Theorem 5.1 one can obtain the following

Corollary 6.1 *Let $f : M \to \mathbb{R}^n$ be an m-dimensional, complete minimal submanifold. Assume that the universal covering M' has finite total scalar curvature. Then, for any fixed $o \in M$,*

$$|\pi_1(M)| \leq \lim_{R\to+\infty} \frac{\mathrm{Vol}\mathbf{B}_R(0)}{\mathrm{Vol}B_R(o)} < +\infty.$$

We point out that a similar conclusion occurs in the (apparently) completely different setting of locally conformally flat manifolds. Recall that an **immersion** $f : (M, \langle,\rangle_M) \to (N, \langle,\rangle_N)$ is said to be **conformal** if the induced metric satisfies $f^* \langle,\rangle_N = \lambda^2 \langle,\rangle_M$ for some $0 < \lambda \in C^\infty(M)$. Say that the m-dimensional Riemannian manifold $(M, \langle,\rangle)$ is **Locally Conformally Flat** (LCF for short) if every point $x \in M$ has a coordinate chart (U, ξ) such that $\xi : U \to \mathbb{R}^m$ is a conformal immersion. Clearly, if M has a global conformal immersion into the equidimensional sphere $\mathbb{S}^m$ then M is LCF. Furthermore, every Riemannian covering of a LCF manifold is itself LCF. A difficult result by G. Tian and J. Viaclovsky, [46], [47], when combined with nice remarks by G. Carron and M. Herzlich, [11], and deep works by N. Kuiper, [25], and R. Schoen and S.T. Yau, [45], implies the following[1]

Theorem 6.2 *Let $(M, \langle,\rangle)$ be a complete, simply connected, LCF manifold of dimension $m \geq 4$. Assume that $Scal = 0$ and $|Ric| \in L^{m/2}(M)$. Then, for any origin $o \in M$, $\mathrm{Vol}B_R(o) \preccurlyeq r^m$.*

[1] more precisely: Tian-Viaclovsky proved their volume estimates assuming (a) the validity of an Euclidean Sobolev inequality, (b) $|Ric_-| \in L^{m/2}$ and (c) $\sup_{M\setminus B_R} |Riem| = o(R^{-2})$. Carron-Hezlich observed that, for a LCF manifold with $Scal = 0$ and satisfying an Euclidean Sobolev inequality, the uniform estimate (c) follows from the integral condition (b). Finally, according to Kuiper, a simply connected LCF manifold has a conformal immersion into $\mathbb{S}^m$ and, by Schoen-Yau, if we further assume that $Scal \leq 0$, then M enjoys the desired Sobolev inequality. See also Section 6.3 below.

On the other hand, as a consequence of results by Schoen-Yau, Carron and K. Akutagawa (see Section 6.3 below) and Carron-Herzlich, we also know that a complete, conformally immersed manifold $f : M^m \to \mathbb{S}^m$ with $Scal = 0$ and $|Ric| \in L^{m/2}(M)$ has a finite number of (asymptotically Euclidean) ends and satisfies the volume condition $r^m \preccurlyeq \mathrm{Vol}B_R(o)$[2]. Thus, applying Theorem 6.2 to the universal covering of M, we can deduce the following version of Corollary 6.1.

Corollary 6.3 *Let $f : M^m \to \mathbb{S}^m$ be a complete, conformally immersed manifold of dimension $m \geq 4$. Assume that $Scal = 0$ and $|Ric| \in L^{m/2}(M)$. Then, for any fixed $o \in M$,*

$$|\pi_1(M)| \leq \lim_{R\to+\infty} \frac{\mathrm{Vol}\mathbf{B}_R(0)}{\mathrm{Vol}B_R(o)} < +\infty.$$

6.2 Asymptotically non-negative Ricci curvature

Both Švarc-Milnor proof of Theorem 3.1 and Anderson proof of Theorem 5.1 use the growth rate of the universal covering M' at the fixed point o' and the growth rate of M at $o = P(o')$. No uniform condition with respect to the centers of the balls is needed. It follows that, potentially, these results could be extended to other curvature bounds depending on the distances from the fixed origins. For instance, the most natural candidate is $Ric(x) \geq -(m-1)G(r_o(x))$ where G is a positive, non-increasing function satisfying

$$\int_0^{+\infty} tG(t)\,dt = b < +\infty.$$

In this situation the Ricci curvature of M is said to be **asymptotically non-negative**. The Bishop-Gromov comparison states that for a complete manifold of asymptotically non-negative curvature, $\mathrm{Vol}B_R(o) \leq e^{(m-1)b}\mathrm{Vol}\mathbf{B}_R(0)$; [52], [37]. Now, from the equation $Ric'(x') = Ric(P(x'))$ we deduce that $Ric'(x') \geq -(m-1)G(r_o(P(x')))$. However, since $-G(t)$ is increasing and $r_{o'}(x') \geq r_o(P(x'))$, we have $-G(r_{o'}(x')) \geq -G(r_o(x))$. Therefore, we are not able to conclude that also M' has asymptotically non-negative curvature. Actually, the above discussion shows that radial curvature conditions like $Ric \geq -(m-1)G(r)$ with G increasing pass from the base to the

[2]again, a little explanation is needed: by Schoen-Yau, since M has a conformal immersion into $\mathbb{S}^m$ and $Scal = 0$, then M enjoys a Sobolev inequality. By Carron-Herzlich, in the full assumptions on M, the manifold has only a finite number of ends. Finally, the Sobolev inequality always implies a lower control on volumes according to results by Carron and Akutagawa, as explained in Section 6.3.

covering manifold. Conversely, if G is decreasing, these curvature conditions descend from the covering manifold to the base.

The following is a kind of (unsatisfactory) generalization of Anderson-Li result. Recall that an m-**dimensional model manifold** is a geodesically complete manifold diffeomorphic to $\mathbb{R}^m$ which is defined by

$$(M, \langle , \rangle) = \left([0, +\infty) \times \mathbb{S}^{m-1}, dr \otimes dr + \sigma(r)^2 d\theta^2 \right)$$

where $d\theta^2$ denotes the stadard metric of $\mathbb{S}^{m-1}$ and $\sigma : [0, +\infty) \to \mathbb{R}$ is a smooth function satisfying the following conditions

$$\begin{cases} (a) & \sigma(r) > 0 \quad \forall r > 0 \\ (b) & \sigma^{(2k)}(0) = 0 \quad \forall k \geq 0 \\ (c) & \sigma'(0) = 1. \end{cases}$$

Conditions (b) and (c) insure that, actually, the Riemannian metric $dr \otimes dr + \sigma(r)^2 d\theta^2$ of $(0, +\infty) \times \mathbb{S}^{m-1}$ extends smoothly at the origin o. The r-coordinate represents the distance from the pole o of the model. Furthermore, for any unit vector field $E \perp \partial r$, the corresponding radial sectional curvature at $x \in M$ is given by

$$Sec_x(\partial r \wedge E) = -\frac{\sigma''}{\sigma}(r(x)).$$

For instance, the standard spaceforms of constant sectional curvature $k \in \mathbb{R}$ are obtained with the choices

$$\sigma(r) = \begin{cases} \frac{1}{\sqrt{k}} \sin\left(\sqrt{k} r\right) & \text{if } k > 0 \\ r & \text{if } k = 0 \\ \frac{1}{\sqrt{-k}} \sinh\left(\sqrt{-k} r\right) & \text{if } k < 0. \end{cases}$$

Here is the announced result. The reader should check the details.

Proposition 6.4 *Let $(M, \langle , \rangle)$ be a complete Riemannian manifold of dimension m. Assume that the Riemannian universal covering M' satisfies $Ric'(x') \geq -(m-1) G(r_{o'}(x'))$, for some origin $o' \in M'$ and for some positive, non-increasing function $G(t)$ such that $tG(t) \in L^1(+\infty)$. Let $\mathbf{M}^m(-G)$ denote the m-dimensional model manifold with radial sectional curvature $-G$ and let $\mathbf{B}_R^{\mathbf{M}}(0)$ be the ball in $\mathbf{M}^m(-G)$ centered at the pole. If $r^m \preccurlyeq \mathrm{Vol} B_r^M(o)$, then*

$$|\pi_1(M)| \leq \lim_{r \to +\infty} \frac{\mathrm{Vol}\mathbf{B}_r^{\mathbf{M}}(0)}{\mathrm{Vol}B_r^M(o)},$$

and the equality holds if and only if M is isometric to a finite quotient of $\mathbf{M}^m(-G)$.

6.3 Sobolev inequalities and π_1

It is well known that the Euclidean space $\mathbb{R}^m$, $m \geq 3$, enjoys the family of Sobolev inequalities

$$(S_p) \qquad \|\varphi\|_{L^{\frac{m}{m-p}}} \leq S \|\nabla\varphi\|_{L^p}$$

for every $1 \leq p < m$, for an explicit (sharp) constant $S = S(m,p) > 0$ and for every $\varphi \in C_c^\infty(\mathbb{R}^m)$. On noting that the geometric analysis on the Euclidean space is greatly simplified by the validity of these powerful inequalities one is naturally led to investigate which complete manifolds enjoy (some of the) (S_p); see e.g. the lecture notes [22] by E. Hebey. For instance, it is known from classical work by J. Michael and L. Simon, [34], that these inequalities (with a worse constant) are inherited by every minimal submanifold of $\mathbb{R}^m$ [3]. According to celebrated works by N. Kuiper and, R. Schoen and S.T. Yau, [45], the sharp Euclidean Sobolev inequality (S_p), $p \geq 2$, is satisfied by every m-dimensional, simply connected, LCF Riemannian manifold M with $Scal_M \leq 0$[4]. In a different direction, according to N. Varopoulos, [48], the entire family of inequalities (S_p) is enjoyed by any complete manifold M with $Ric \geq 0$ and maximal volume growth $\mathrm{Vol}B_R(o) \geq CR^m$. It is important to observe that the lower volume estimate is always a necessary condition for the validity of (S_p)[5]. Indeed we have the following result due to G. Carron, [7], and K. Akutagawa, [1]. See also [39].

Theorem 6.5 *Let $(M, \langle,\rangle)$ be a complete manifold satisfying (S_p) for some $p \geq 1$. Then, there exists a (small and explicit) constant $C > 0$ such that, for every origin $o \in M$, $\mathrm{Vol}B_R(o) \geq CR^m$.*

In particular, using Theorem 6.5 together with Anderson-Li estimating result, we see that the situation considered by Varopoulos imposes the following topological restrictions on the underlying manifold.

Corollary 6.6 *Let $(M, \langle,\rangle)$ be a complete manifold of dimension $m \geq 3$ satisfying $Ric \geq 0$. If M enjoys (S_p) for some $p \geq 1$, then $|\pi_1(M)| < +\infty$.*

[3] actually, (S_p) is enjoyed by every complete, immersed submanifold $f: M \to N$ into the Cartan-Hadamard manifold N provided the mean curvature vector filed $\mathbf{H}$ of f satisfies $\|\mathbf{H}\|_{L^{m/2}} < +\infty$; see [23] and [8].

[4] the assumption $\pi_1(M) = 1$ can be replaced by the requirement that M has a conformal immersion into the equidimensional standard sphere $\mathbb{S}^m$.

[5] actually, Varopoulos proved that, under the assumption $Ric \geq 0$, the validity of (S_p) is equivalent to the maximal volume growth condition.

Example 6.7 *The assumption "Ric ≥ 0" cannot be relaxed to "Ric ≥ 0 outside a compact set". Indeed, consider the complete manifold $M = \mathbb{R}^3 \# \mathbb{T}^3$. By Seifert-Van Kampen theorem, $\pi_1(M) \simeq \pi_1(\mathbb{R}^3) * \pi_1(\mathbb{T}^3) \simeq \mathbb{Z}^3$. On the other hand, outside a sufficiently large compact set, say K, $\langle , \rangle = \mathrm{can}_{\mathbb{R}^3}$. In particular, (a) $Ric_M = 0$ on $M \backslash K$ and (b) the Sobolev inequality (S_p) holds on $M \backslash K$. To conclude that, in fact, the Sobolev inequality extends (with a different constant) on all of M we use the following deep result by G. Carron, [7], recently extended to every $p > 1$ in [39].*

Theorem 6.8 *Let $(M, \langle , \rangle)$ be a complete manifold of dimension $m \geq 3$. If (S_p) is satisfied outside some compact set $K \subset M$ then the same Sobolev inequality, but possibly with a different constant, holds on all of M.*

We conclude our (very) brief tour into Euclidean Sobolev inequalities by noting that Corollary 6.6 can assume the next more general form. This can be considered as a modified version of Anderson-Li result and follows by putting together a certain number of the previous remarks in the Chapter.

Theorem 6.9 *Let $(M, \langle , \rangle)$ be a complete manifold of dimension $m \geq 3$, satisfying the Sobolev inequality (S_p) outside some compact set. If the universal covering $(M', \langle , \rangle')$ of M satisfies the volume growth condition $\mathrm{Vol} B'_R \leq C R^m$, then $|\pi_1(M)| < +\infty$.*

6.4 Cardinalities of π_1 in Anderson-Li result

According to Theorem 5.1 a complete manifold with non-negative Ricci curvature and maximal volume growth has finite fundamental group. Variations of this result have been pointed out in the previous sections. It is then natural to ask whether or not all possible cardinalities of the fundamental group can be achieved[6]. If we consider the "Sobolev-version" of Theorem 5.1 stated in Theorem 6.9, it is quite easy to show that the answer is yes.

Example 6.10 *Fix $k \geq 2$ and consider the complete, 3-dimensional manifold $M = \mathbb{R}^3 \# \mathbb{L}(k,1)$ where $\mathbb{L}(k,1)$ denotes the lens space introduced in Example 1.6. Then, $\pi_1(M) \simeq \pi_1(\mathbb{R}^3) * \pi_1(\mathbb{L}(k,1)) \simeq \mathbb{Z}_k$. In particular, the Riemannian universal covering $P : M' \to M$ of M is a k-fold covering. Since $\langle , \rangle_M = \mathrm{can}_{\mathbb{R}^3}$ outside a compact set K, then the Sobolev inequality (S_p) holds on $M \backslash K$. Moreover, we have that $Ric_{M'} = 0$ outside*

[6] the question was raised by Daniele and Giona. This section reports (part of) the subsequent discussions

the compact set $P^{-1}(K)$. Therefore, by Bishop-Gromov comparison (e.g. for manifolds with asymptotically non-negative Ricci curvature, [52], [37]) the volume growth of M' satisfies $\mathrm{Vol}B'_R \leq CR^m$. For every $k \geq 1$, we have thus construct a Riemannian manifold M satisfying all the assumptions of Theorem 6.9 and such that $|\pi_1(M)| = k$.

In the assumptions of Theorem 5.1 answering the question is much more difficult. However we have the following important result by Y. Otsu, [35] Theorem 3.

Theorem 6.11 *Let $m \geq 6$. Then, for every $k, h \in \mathbb{N} \setminus \{0\}$ with $\mathrm{MCD}(k,h) = 1$, there exists a Riemannian metric on $M = \mathbb{R}^{m-3} \times \mathbb{L}(k,h)$ with non-negative Ricci curvature and maximal volume growth.*

Actually Otsu result is originally stated with $\mathbb{S}^l$ instead of the lens space. However, a quick inspection of the proof shows that the unit sphere can be replaced by any one of its isometric quotients. Now, since $\pi_1(M) \simeq \pi_1(\mathbb{R}^{m-3}) \times \pi_1(\mathbb{L}(k,h)) \simeq \mathbb{Z}_k$ it follows from Otsu result that also in the assumptions of Theorem 5.1, $\pi_1(M)$ can achieve all possible cardinalities.

Appendix A

The Bochner formula and its friends

In this appendix we collect some key tools of geometric analysis that we have repeatedly used in the previous sections: an integral estimate of the Ricci tensor along minimizing geodesics, the Laplacian and the volume comparison theorems. It happens that these results have their common root in the celebrated Bochner formula for smooth, real valued functions. Thus, to begin with, we derive such a formula. Our reference for Riemannian geometry is Petersen's book, [36].

Theorem A.1 (Bochner formula) *Let $(M, \langle , \rangle)$ be a Riemannian manifold and let $u \in C^{\infty}(M)$. Then*

$$\frac{1}{2}\Delta |\nabla u|^2 = |\mathrm{Hess}\,(u)|^2 + \langle \nabla \Delta u, \nabla u \rangle + Ric\,(\nabla u, \nabla u)\,.$$

Proof. Fix a normal coordinate system centered at a point x. The corresponding components of du, $\mathrm{Hess}\,(u) = Ddu$ and $D\mathrm{Hess}\,(u)$ are denoted by u_i, u_{ij} and u_{ijk}, respectively. Note that,

$$u_{ij} = u_{ji}, \quad u_{ijk} = u_{jik}$$

and, at $x(!)$, we have[1]

$$\begin{array}{ll} \Delta u = \sum_i u_{ii}, & |\mathrm{Hess}\,(u)|^2 = \sum_{i,j} u_{ij}^2 \\ \langle \nabla \Delta u, \nabla u \rangle = \sum_{i,k} u_{iik} u_k, & |\nabla u|^2 = \sum_i u_i^2. \end{array}$$

[1] with respect to this coordinate system, the local computations at the fixed point x can be done as in $\mathbb{R}^m$. The curvature tensor enters the game when we commute third (or higher) derivatives.

Now, direct computations show that, at x,

$$\begin{aligned}\frac{1}{2}\Delta\left|\nabla u\right|^{2} &= \sum_{i,j}u_{ij}^{2}+\sum_{i,j}u_{ijj}u_{i}\\ &= \left|\mathrm{Hess}\left(u\right)\right|^{2}+\sum_{i,j}u_{jij}u_{i}.\end{aligned}$$

In order to deal with the last term, we recall the commutation rule

$$u_{ijk}-u_{ikj}=-R_{lijk}\partial_{l}u,$$

where the R_{ijkl}'s denote the components of the Riemann tensor. Tracing these relations at x, with respect to i and k, we deduce

$$\sum_{k}u_{kjk}=\sum_{k}u_{kkj}+\sum_{l}Ric_{jl}u_{l},$$

and inserting into the above we conclude

$$\begin{aligned}\frac{1}{2}\Delta\left|\nabla u\right|^{2} &= \left|\mathrm{Hess}\left(u\right)\right|^{2}+\sum_{i,k}u_{kki}u_{i}+\sum_{j,l}Ric_{jl}u_{l}u_{j}\\ &= \left|\mathrm{Hess}\left(u\right)\right|^{2}+\left\langle\nabla\Delta u,\nabla u\right\rangle+Ric\left(\nabla u,\nabla u\right).\end{aligned}$$

■

The next inequalities are usually obtained via the second variation formula for the arc-length and Jacobi fields. However, they can be derived directly from the Bochner formula applied to the distance function; see [40] and e.g. [38].

Lemma A.2 *Let $(M,\langle,\rangle)$ be a Riemannian manifold. Fix $o\in M$ and let $r\left(x\right)=d\left(x,o\right)$. For any point $q\in M$, let $\gamma_{q}:\left[0,r\left(q\right)\right]\to M$ be a minimizing geodesic from o to q such that $\left|\dot{\gamma}_{q}\right|=1$. If $h\in Lip_{loc}\left(\mathbb{R}\right)$ is such that $h\geq0$ and $h\left(0\right)=0$, then, for every $q\notin\mathrm{cut}(o)$,*

$$h^{2}\left(r\left(q\right)\right)\Delta r(q)\leq\left(m-1\right)\int_{0}^{r(q)}\left(h'\right)^{2}ds-\int_{0}^{r(q)}h^{2}Ric\left(\dot{\gamma}_{q},\dot{\gamma}_{q}\right)ds. \tag{A.1}$$

If in addition $h\left(r\left(q\right)\right)=0$, then for every $q\in M$,

$$0\leq\left(m-1\right)\int_{0}^{r(q)}\left(h'\right)^{2}ds-\int_{0}^{r(q)}h^{2}Ric\left(\dot{\gamma}_{q},\dot{\gamma}_{q}\right)ds. \tag{A.2}$$

Proof. Fix a point $q \in M$ and suppose that $q \notin \mathrm{cut}(o)$. Applying the Bochner formula to $u = \nabla r$ we deduce

$$0 = |\mathrm{Hess}\,(r)|^2 + \langle \nabla \Delta r, \nabla r \rangle + Ric\,(\nabla r, \nabla r)\,.$$

By the Cauchy-Schwarz inequality, on noting also that $\mathrm{Hess}\,(r)\,(\nabla r, \cdot) = 0$, it follows that

$$|\mathrm{Hess}\,(r)|^2 \geq \frac{(\Delta r)^2}{m-1}.$$

Inserting into the above gives

$$\frac{(\Delta r)^2}{m-1} + \langle \nabla \Delta r, \nabla r \rangle + Ric\,(\nabla r, \nabla r) \leq 0. \tag{A.3}$$

Since the minimal geodesic γ_q is an integral curve of ∇r we can evaluate (A.3) along γ_q to get

$$\frac{(\Delta r \circ \gamma_q)^2}{m-1} + \frac{d}{ds}\,(\Delta r \circ \gamma_q) + Ric\,(\dot{\gamma_q}, \dot{\gamma_q}) \leq 0. \tag{A.4}$$

If $h \in Lip_{loc}\,(\mathbb{R})$ is such that $h \geq 0$, $h\,(0) = 0$, multiplying by h^2 equation (A.4) and integrating on $[0, t]$ we get

$$\int_0^t h^2 \frac{(\Delta r \circ \gamma_q)^2}{m-1} ds + \int_0^t \frac{d}{ds}\,(\Delta r \circ \gamma_q)\, h^2 ds + \int_0^t h^2 Ric\,(\dot{\gamma_q}, \dot{\gamma_q})\, ds \leq 0.$$

Integrating by parts and noting that $(\Delta r \circ \gamma_q)\, h^2 \to 0$ as $r \to 0$ we obtain

$$\begin{aligned} 0 \geq & \int_0^t h^2 \frac{(\Delta r \circ \gamma_q)^2}{m-1} ds + h^2\,(t)\, \Delta r \circ \gamma_q\,(t) \\ & - 2\int_0^t hh'\,(\Delta r \circ \gamma_q)\, ds + \int_0^t Ric\,(\dot{\gamma_q}, \dot{\gamma_q})\, h^2 ds. \end{aligned}$$

Since

$$\begin{aligned} 2hh'\,(\Delta r \circ \gamma_q) =& \frac{2h \Delta r \circ \gamma_q}{\sqrt{m-1}} \sqrt{m-1}\, h' \\ \leq & \frac{h^2\,(\Delta r \circ \gamma_q)^2}{m-1} + (m-1)\,(h')^2\,, \end{aligned}$$

we deduce that

$$0 \geq h^2\,(t)\, \Delta r \circ \gamma_q\,(t) - (m-1) \int_0^t (h')^2\, ds + \int_0^t h^2 Ric\,(\dot{\gamma_q}, \dot{\gamma_q})\, ds$$

and, setting $t = r(q)$, we conclude that

$$h^2(r(q))\,\Delta r(q) \leq (m-1)\int_0^{r(q)} (h')^2\,ds - \int_0^{r(q)} h^2 Ric(\dot{\gamma}_q, \dot{\gamma}_q)\,ds.$$

If in addition h satisfies $h^2(r(q)) = 0$, then the above inequality becomes

$$0 \leq (m-1)\int_0^{r(q)} (h')^2\,ds - \int_0^{r(q)} h^2 Ric(\dot{\gamma}_q, \dot{\gamma}_q)\,ds. \tag{A.5}$$

This completes the proof if $q \notin \mathrm{cut}(o)$. In the general case inequality (A.5) can be extended to any $q \in M$ using the Calabi trick. Indeed, suppose that $q \in \mathrm{cut}(o)$. Translating the origin o to $o_\varepsilon = \gamma_q(\varepsilon)$ so that $q \notin \mathrm{cut}(o_\varepsilon)$, using the triangle inequality and, finally, taking the limit as $\varepsilon \to 0$, one checks that (A.5) holds also in this case. ■

Under suitable assumptions on the Ricci curvature, a direct use of inequality (A.1) with an appropriate choice of the function h implies the next important comparison result. We recall from Definition 3.19 and Remark 3.20 that, given $u \in W^{1,2}_{loc}(M)$ and $f \in L^1_{loc}(M)$, the inequality

$$\Delta u \leq f, \text{ weakly on } M$$

means that, for every $0 \leq \varphi \in C^\infty_c(M)$, it holds

$$\int_M u\Delta\varphi = -\int_M \langle \nabla u, \nabla \varphi \rangle \leq \int_M f\varphi.$$

Theorem A.3 (Laplacian comparison) *Let $(M, \langle , \rangle)$ be a complete manifold of dimension m and let $r(x) = d(x, o)$ be the distance function from a fixed origin $o \in M$. Assume that the radial Ricci curvature satisfies $Ric(\nabla r, \nabla r) \geq (m-1)c$ with $c \in \mathbb{R}$. Set*

$$\mathrm{sn}_c(t) = \begin{cases} \frac{1}{\sqrt{c}} \sin(\sqrt{c}t) & \text{if } c > 0 \\ t & \text{if } c = 0 \\ \frac{1}{\sqrt{-c}} \sinh(\sqrt{-c}t) & \text{if } c < 0. \end{cases}$$

Then,

$$(\Delta r)(x) \leq (m-1)\frac{\mathrm{sn}_c'(r(x))}{\mathrm{sn}_c(r(x))} \tag{A.6}$$

holds pointwise on $M \backslash \mathrm{cut}(o)$ and weakly on all of M. Moreover, the equality holds on some ball $B_{R_0}(o)$ if and only if $B_{R_0}(o)$ is isometric to the ball $\mathbf{B}_{R_0}(0)$ in the corresponding space-form of $\mathbf{M}^m(c)$ of constant curvature c.

Proof. Assume that $x \notin \mathrm{cut}(o)$. Using the Ricci curvature assumption into (A.1) we get

$$h^2(r(x))\,\Delta r(x) \le (m-1)\int_0^{r(x)} (h')^2\,ds - (m-1)\,c\int_0^{r(x)} h^2 ds \tag{A.7}$$
$$= (m-1)\int_0^{r(x)} \left\{ (h')^2 - ch^2 \right\} ds.$$

Whence, choosing $h(s) = \mathrm{sn}_c(s)$ we deduce the validity of (A.6) on $M\backslash \mathrm{cut}(o)$.

Although $M\backslash\mathrm{cut}(o)$ is an open set of full measure, the extension of (A.6) to all of M (in the sense of distributions) is nontrivial, [51], [10]. See also [37]. Let E_o be the maximal domain of definition of normal coordinates at o; thus $\exp_o|_{E_0}$ is a diffeomorphism onto its image and we have $\mathrm{cut}(o) = \partial(\exp_o(E_o))$. Since E_o is a star-shaped domain, we can exhaust E_o by a family $\{E_o^n\}$ of relatively compact, star-shaped domains with smooth boundary. We set $\Omega^n = \exp_o(E_o^n)$ so that

$$\overline{\Omega}^n \subset \Omega^{n+1} \text{ and } \cup_n \Omega^n = \Omega.$$

The fact that each E_o^n is star-shaped implies

$$\frac{\partial r}{\partial \nu_n} > 0, \quad \text{on } \partial\Omega^n \tag{A.8}$$

where ν_n denotes the out-ward unit normal to $\partial\Omega^n$. Now, from the first part of the proof, since $r \in C^\infty(\Omega^n\backslash\{o\})$, we have

$$(\Delta r)(x) \le (m-1)\frac{\mathrm{sn}_c'(r(x))}{\mathrm{sn}_c(r(x))} \quad \text{pointwise on } \Omega^n\backslash\{o\}. \tag{A.9}$$

Let $0 \le \varphi \in C_c^\infty(\Omega^n)$. We claim that, for every n,

$$-\int_{\Omega^n} \langle \nabla r, \nabla\varphi\rangle \le (m-1)\int_{\Omega^n} \frac{\mathrm{sn}_c'(r(x))}{\mathrm{sn}_c(r(x))}\varphi. \tag{A.10}$$

Indeed, fix $\delta > 0$ small and apply the divergence theorem on $\overline{\Omega^n}\backslash B_\delta(o)$ to obtain

$$-\int_{\Omega^n\backslash B_\delta(o)} \langle\nabla r, \nabla\varphi\rangle = \int_{\Omega^n\backslash B_\delta(o)} \varphi\Delta r - \int_{\partial\Omega^n\cup\partial B_\delta(o)} \varphi\frac{\partial r}{\partial\nu_n} \tag{A.11}$$

where ν_n is the out-ward unit normal to $\partial\Omega^n \cup \partial B_\delta(o)$. Using (A.8), (A.9) and (A.11), we obtain

$$-\int_{\Omega^n\backslash B_\delta(o)} \langle\nabla r, \nabla\varphi\rangle \le (m-1)\int_{\Omega^n\backslash B_\delta(o)} \frac{\mathrm{sn}_c'(r(x))}{\mathrm{sn}_c(r(x))}\varphi + \int_{\partial B_\delta(o)} \varphi.$$

Whence, letting $\delta \to 0$ we conclude the validity of the claimed inequality (A.10). Now, since $M = \Omega \cup \mathrm{cut}(o)$ and $\mathrm{cut}(o)$ has measure 0, letting $n \to +\infty$ in (A.10) shows that

$$-\int_M \langle \nabla r, \nabla \varphi \rangle \leq (m-1) \int_M \frac{\mathrm{sn}_c'(r(x))}{\mathrm{sn}_c(r(x))} \varphi.$$

Now, suppose that in (A.6) the equality sign holds on some ball $B_{R_0}(o)$. Since the function $\mathrm{sn}_c'(r(x)) / \mathrm{sn}_c(r(x))$ is locally Lipschitz, by elliptic regularity $r(x)$ is smooth on $B_{R_0}(o) \setminus \{o\}$. By a result of R. Bishop, [4], [49], this implies that $\mathrm{cut}(o) \cap B_{R_0}(o) = \emptyset$ and the equality in (A.6) holds pointwise in $B_{R_0}(o)$. In particular, we have equality in (A.7) which, in turn, forces

$$Ric(\nabla r, \nabla r) = (m-1)\, c$$

and

$$|\mathrm{Hess}\,(r)|^2 = \frac{(\Delta r)^2}{m-1}, \text{ on } B_{R_0}(o)\,;$$

see the proof of Lemma A.2. By the equality case in the Cauchy-Schwarz inequality, we have

$$\mathrm{Hess}\,(r) = \lambda\left(\langle , \rangle - dr \otimes dr\right), \text{ on } B_{R_0}(o)$$

for some function $\lambda(x)$. Since

$$\Delta r = (m-1) \frac{\mathrm{sn}_c'}{\mathrm{sn}_c} \text{ in } B_{R_0}(o)\,,$$

tracing the above equation shows that

$$\lambda(x) = \frac{\mathrm{sn}_c'(r(x))}{\mathrm{sn}_c(r(x))}.$$

We have thus proved that, in $B_{R_0}(o)$,

$$\mathrm{Hess}\,(r) = \frac{\mathrm{sn}_c'(r)}{\mathrm{sn}_c(r)}\left(\langle , \rangle - dr \otimes dr\right).$$

To conclude, we can now apply the following general result. ∎

Lemma A.4 *Let $(M, \langle , \rangle)$ be a complete Riemannian manifold. Assume that $B_R(o) \cap \mathrm{cut}(o) = \emptyset$. If*

$$\mathrm{Hess}\,(r) = \frac{\mathrm{sn}_c'(r)}{\mathrm{sn}_c(r)}\left(\langle , \rangle - dr \otimes dr\right), \text{ in } B_R(o) \tag{A.12}$$

then $B_R(o)$ is isometric to the ball $\mathbf{B}_{R_0}(0)$ in the corresponding space-form $M^m(c)$ of constant curvature c.

Proof. Introduce geodesic polar coordinates (r,θ) centered at o. Furthermore, let $\{\theta^{\alpha}\}$ be a local orthonormal coframe for the standard metric of $\mathbb{S}^{m-1}$ and extend it radially. The local (radially extended) dual frame is denoted by $\{E_{\alpha}\}$. Then, by Gauss lemma, the Riemannian metric of M writes as

$$\langle,\rangle = dr\otimes dr + \sigma_{\alpha\beta}\left(r,\theta\right) d\theta^{\alpha}\otimes d\theta^{\beta}.$$

Note that, since every Riemannian metric is asymptotically Euclidean,

$$\sigma_{\alpha\beta}\left(r,\theta\right) = r^{2}\delta_{\alpha\beta} + o\left(r^{2}\right),$$

as $r\to 0$. We show that

$$\sigma_{\alpha\beta}\left(r,\theta\right) = \delta_{\alpha\beta}\mathrm{sn}_{c}\left(r\right). \tag{A.13}$$

Indeed, according to (A.12), taking the Lie derivative of $\langle,\rangle$ in direction ∇r gives

$$L_{\nabla r}\left\langle,\right\rangle\left(E_{\alpha},E_{\beta}\right) = \mathrm{Hess}\left(r\right)\left(E_{\alpha},E_{\beta}\right) = \frac{\mathrm{sn}_{c}'\left(r\right)}{\mathrm{sn}_{c}\left(r\right)}\delta_{\alpha\beta}.$$

On the other hand

$$L_{\nabla r}\left\langle,\right\rangle\left(E_{\alpha},E_{\beta}\right) = \frac{\partial\sigma_{\alpha\beta}}{\partial r}.$$

Thus,

$$\begin{cases} \dfrac{\partial\sigma_{\alpha\beta}}{\partial r} = \dfrac{\mathrm{sn}_{c}'\left(r\right)}{\mathrm{sn}_{c}\left(r\right)}\delta_{\alpha\beta} \\ \sigma_{\alpha\beta}\left(r,\theta\right) = r^{2}\delta_{\alpha\beta} + o\left(r^{2}\right), \end{cases}$$

and integrating we deduce the validity of (A.13). ■

We conclude the section by proving the volume comparison by Bishop and Gromov. It is obtained as an application of the Laplacian comparison, [37].

Theorem A.5 (Bishop-Gromov) *Let* $(M,\langle,\rangle)$ *be a complete Riemannian manifold of dimension* $\dim M = m$ *satisfying*

$$Ric\left(\nabla r_{o},\nabla r_{o}\right) \geq \left(m-1\right)c, \text{ on } M$$

for some constant $c\in\mathbb{R}$ *and for some reference origin* $o\in M$*. Denote by* $\mathbf{B}_{r}\left(0\right)$ *the geodesic ball of radius* $r>0$ *and centered at the origin of the* m*-dimensional space-form* $\mathbf{M}^{m}\left(c\right)$ *of constant sectional curvature* c*. Then, the functions*

$$r\longmapsto\frac{\mathrm{Area}\partial B_{r}(o)}{\mathrm{Area}\partial\mathbf{B}_{r}(o)} \quad \text{and} \quad r\longmapsto\frac{\mathrm{Vol}B_{r}(o)}{\mathrm{Vol}\mathbf{B}_{r}(o)}$$

are non-increasing. In particular,

$$\mathrm{Vol} B_r(o) \leq \mathrm{Vol}\mathbf{B}_r(0)\ ,\ \forall r > 0$$

and the equality holds for some R_0 if and only if $B_{R_0}(o)$ is isometric to $\mathbf{B}_{R_0}(0)$.

Proof. Observe that

$$\mathrm{Area}\partial\mathbf{B}_r(o) = \omega_{m-1}\mathrm{sn}_c(r)^{m-1}\,,$$

and

$$\mathrm{Vol}\mathbf{B}_r(0) = \omega_{m-1}\int_0^r \mathrm{sn}_c(t)^{m-1}\,dt,$$

where we have set

$$\omega_{m-1} = \mathrm{Vol}\left(\mathbb{S}^{m-1}\right).$$

We prove that

$$R \mapsto \frac{\mathrm{Area}\partial B_R(o)}{\mathrm{sn}_c(R)^{m-1}} \quad \text{is non increasing.} \tag{A.14}$$

This latter, in turn, implies that

$$R \mapsto \frac{\mathrm{Vol}B_R V\,(o)}{\int_0^R \mathrm{sn}_c V\,(t)^{m-1}\,dt} \quad \text{is non increasing.} \tag{A.15}$$

Indeed, by the co-area formula,

$$\mathrm{Vol}B_R(o) = \int_0^R \mathrm{Area}\partial B_t(o)\,dt,$$

and it was observed by Gromov that, for general real valued functions $f\,(t) \geq 0$, $g\,(t) > 0$,

$$t \to \frac{f\,(t)}{g\,(t)} \text{ is decreasing} \implies t \to \frac{\int_0^t f}{\int_0^t g} \text{ is decreasing.}$$

Let us prove (A.14). In case o is a pole of M one simply integrates the radial vector field

$$X = \mathrm{sn}_c(r\,(x))^{1-m}\,\nabla r$$

on concentric balls $B_R\,(o)$, and uses the divergence and the Laplacian comparison theorems. However, in general, objects are non-smooth and inequalities are intended in the sense of distributions. Therefore, we have to take some extra care.

The Laplacian comparison theorem asserts that

$$\Delta r(x) \leq (m-1)\frac{\mathrm{sn}_c'(r(x))}{\mathrm{sn}_c(r(x))} \tag{A.16}$$

pointwise on the open, star-shaped, full-measured set $M \setminus \mathrm{cut}(o)$ and weakly on all of M. Thus, for every $0 \leq \varphi \in Lip_c(M)$,

$$-\int \langle \nabla r, \nabla \varphi \rangle \leq (m-1)\int \frac{\mathrm{sn}_c'(r(x))}{\mathrm{sn}_c(r(x))}\varphi. \tag{A.17}$$

For any $\varepsilon > 0$, consider the radial cut-off function

$$\varphi_\varepsilon(x) = \rho_\varepsilon(r(x))\,\mathrm{sn}_c(r(x))^{-m+1} \tag{A.18}$$

where ρ_ε is the piecewise linear function

$$\rho_\varepsilon(t) = \begin{cases} 0 & \text{if } t \in [0, r) \\ \frac{t-r}{\varepsilon} & \text{if } t \in [r, r+\varepsilon) \\ 1 & \text{if } t \in [r+\varepsilon, R-\varepsilon) \\ \frac{R-t}{\varepsilon} & \text{if } t \in [R-\varepsilon, R) \\ 0 & \text{if } t \in [R, \infty). \end{cases} \tag{A.19}$$

Note that

$$\nabla \varphi_\varepsilon = \left\{ -\frac{\chi_{R-\varepsilon,R}}{\varepsilon} + \frac{\chi_{r,r+\varepsilon}}{\varepsilon} - (m-1)\frac{\mathrm{sn}_c'(r)}{\mathrm{sn}_c(r)}\rho_\varepsilon \right\} \mathrm{sn}_c(r)^{-m+1}\,\nabla r,$$

for a.e. $x \in M$, where $\chi_{s,t}$ is the characteristic function of the annulus $B_t(o) \setminus B_s(o)$. Therefore, using φ_ε into (A.17) and simplifying, we get

$$\frac{1}{\varepsilon}\int_{B_R(o)\setminus B_{R-\varepsilon}(o)} \mathrm{sn}_c(r(x))^{-m+1} \leq \frac{1}{\varepsilon}\int_{B_{r+\varepsilon}(o)\setminus B_r(o)} \mathrm{sn}_c(r(x))^{-m+1}.$$

Using the co-area formula we deduce that

$$\frac{1}{\varepsilon}\int_{R-\varepsilon}^{R} \mathrm{Area}\partial B_t(o)\,\mathrm{sn}_c(r(x))^{-m+1} \leq \frac{1}{\varepsilon}\int_{r}^{r+\varepsilon} \mathrm{Area}\partial B_t(o)\,\mathrm{sn}_c(r(x))^{-m+1}$$

and, letting $\varepsilon \searrow 0$,

$$\frac{\mathrm{Area}\partial B_R(o)}{\mathrm{sn}_c(R)^{m-1}} \leq \frac{\mathrm{Area}\partial B_r(o)}{\mathrm{sn}_c(r)^{m-1}}$$

for a.e. $0 < r < R$. This proves the validity of (A.14), hence of (A.15). In particular, for every $R \geq r > 0$,

$$\mathrm{Vol}B_R(o) \leq \frac{\mathrm{Vol}B_r(o)}{\mathrm{Vol}\mathbf{B}_r(0)}\mathrm{Vol}\mathbf{B}_R(0).$$

Since every Riemannian metric is infinitesimally Euclidean,

$$\mathrm{Vol}\mathbf{B}_r(0)\,,\mathrm{Vol}B_r(o)\sim\frac{\omega_m}{m}r^m,\ \text{ as } r\to 0.$$

Therefore, letting $r\to 0$ into the above, we obtain

$$\mathrm{Vol}B_R(o)\leq\mathrm{Vol}\mathbf{B}_R(0)\,.$$

To conclude, the reader should verify that if the equality holds for some $R_0>0$ then $\Delta r=(m-1)\,\mathrm{sn}_c'(r)\,/\mathrm{sn}_c(r)$ on $B_{R_0}(o)$ and the claimed isometry between $B_{R_0}(o)$ and $\mathbf{B}_{R_0}(o)$ follows from the equality case in the Laplacian comparison theorem. ■

Remark A.6 *We note that the comparison for the area functions, when combined with the co-area formula, gives a comparison result for the volumes of concentric annuli. Indeed, let* $A(r,R)=B_R(o)\setminus\overline{B_r(o)}$ *and use the symbol* $\mathbf{A}\,(r,R)$ *for the annulus in the corresponding space-form. Note that, from the Bishop-Gromov comparison, recalling also that the metric is infinitesimally Euclidean, we have*

$$\mathrm{Area}\partial B_t(o)\leq\mathrm{Area}\partial\mathbf{B}_t(0)\,.$$

Therefore

$$\mathrm{Vol}A(r,R)=\int_r^R\mathrm{Area}\partial B_t(o)\,dt\leq\int_r^R\mathrm{Area}\partial\mathbf{B}_t(0)\,dt=\mathrm{Vol}\mathbf{A}(r,R)\,.$$

For instance, assume $Ric\geq 0$*. Then, for any fixed* $k>0$*, we deduce*

$$\mathrm{Vol}A(R,R+k)\preccurlyeq R^{m-1},$$

a fact that has been used in a crucial way in the proof of Theorem 5.1.

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