ON THE DIRICHLET PROBLEM FOR p-HARMONIC MAPS I: COMPACT TARGETS

STEFANO PIGOLA AND GIONA VERONELLI

Abstract. In this paper we solve the relative homotopy Dirichlet problem for p-harmonic maps from compact manifolds with boundary to compact manifolds of non-positive sectional curvature. The proof, which is based on the direct calculus of variations, uses some ideas of B. White to define the relative d-homotopy type of Sobolev maps. One of the main points of the proof consits in showing that the regularity theory by Hardt and Lin can be applied. A comprehensive uniqueness result for general complete targets with non-positive curvature is also given.

1. Introduction

Let (M, g) and (N, h) be Riemannian manifolds of dimensions m and n respectively. Let $u : M \to N$ be a C^1 map. The p-energy density $e_p(u) : M \to \mathbb{R}$ is the non-negative function defined on M as

$$e_p(u)(x) = \frac{1}{p}|du|^p_{HS}(x).$$

Here the differential du is considered as a section of the $(1,1)$-tensor bundle along the map u, i.e. $du \in \Gamma(T^*M \otimes u^{-1}TN)$ is a vector valued differential 1-form. Moreover $T^*M \otimes u^{-1}TN$ is endowed with its Hilbert-Schmidt scalar product. If $\Omega \subset M$ is a compact domain, we define the p-energy of $u|_\Omega : \Omega \to N$ by

$$E^\Omega_p(u) = \int_\Omega e_p(u)dV_M.$$

Let X be a C^1 vector field along u, i.e. a section of the bundle $u^{-1}TN$, supported in Ω. Then

$$u_t(x) = {}^N\exp_{u(x)} tX(x).$$

defines a variation of u which preserves u on $\partial\Omega$. The map $u : M \to N$ is said to be p-harmonic if, for each compact domain $\Omega \subset M$, it is a stationary point of the p-energy functional, that is

$$\left.\frac{d}{dt}\right|_{t=0} E^\Omega_p(u_t) = \int_M \left\langle |du|^{p-2}du, dX\right\rangle_{HS} dV_M = 0.$$

The latter equality corresponds to the weak formulation of the p-laplacian equation

$$\Delta_p u = \operatorname{div}(|du|^{p-2}du) = 0. \tag{1}$$

Date: June 30, 2014.

2010 *Mathematics Subject Classification.* 58E20.

Key words and phrases. p-harmonic maps, Dirichlet problem, d-homotopy classes.

The second author has been partially supported by *INdAM Fellowships in Mathematics and/or Applications for Experienced Researchers cofunded by Marie Curie*.

Here $-\operatorname{div} = \delta$ is the formal adjoint of the exterior differential d, with respect to the standard L^2 inner product on vector-valued differential 1-forms on M. In local coordinates, (1) takes the expression

$$(\Delta_p u)^A = g^{ij}\left(\frac{\partial}{\partial x^j}\left(|du|^{p-2}\frac{\partial u^A}{\partial x^i}\right) - {}^M\Gamma^k_{ij}\frac{\partial u^A}{\partial x^k}|du|^{p-2} + {}^N\Gamma^A_{BC}\frac{\partial u^B}{\partial x^i}\frac{\partial u^C}{\partial x^j}|du|^{p-2}\right) = 0,$$

which, in turn, can be written in the compact form

$$(\Delta_p u)^A = \operatorname{div}\left(|du|^{p-2}\nabla u^A\right) + |du|^{p-2}\Gamma^A(du,du) = 0.$$

It's worthwhile to observe that, in case $N \hookrightarrow \mathbb{R}^q$ is isometrically embedded in some Euclidean space $\mathbb{R}^q$ with second fundamental form $\mathcal{A}$, the above definition of p-harmonicity for a C^1 map $u : M \to \mathbb{R}^q$ is equivalent to the standard notion of weak p-harmonicity, that is

$$\int |Du|^{p-2}\left\{Du \cdot D\varphi + \mathcal{A}\left(Du,Du\right)\cdot\varphi\left(x\right)\right\} = 0, \quad \forall\varphi \in C^\infty_c(\Omega,\mathbb{R}^q) \tag{2}$$

where we have set

$$Du \cdot D\varphi = g^{ij}\delta_{AB}\frac{\partial u^A}{\partial x^i}\frac{\partial \varphi^B}{\partial x^j}$$

and

$$\mathcal{A}\left(Du,Du\right)\cdot\varphi = g^{ij}\delta_{CD}\mathcal{A}^C_{AB}\frac{\partial u^A}{\partial x^i}\frac{\partial u^B}{\partial x^j}\varphi^D.$$

To see this, it's enough to take

$$X_x = D\left(\Pi_N\right)|_{u(x)} \cdot \varphi\left(x\right) \in T_{u(x)}N \subset \mathbb{R}^q,$$

where Π_N is the nearest point projection from a tubular neighborhood of N in $\mathbb{R}^q$.
The theory of p-harmonic maps between Riemannian manifolds and p-energy minimizers has undergone a great development in the last two decades. Among the works on the subject, let us recall for instance [DF, DGK, Fu, Fu2, FR3, Ga, HL, N, XY], dedicated to the regularity theory, and [Wh, DF2, PRS, T, We2, Ma, PV1] which are concerned mostly with the connections to the geometry of the manifolds. Beside these, and in the special case of compact manifolds without boundaries, it is also worth to point out the very recent and interesting developments in the parabolic theory of p-harmonic maps. For instance, we quote [Hu, Mi, FR, FR2].

This paper is the first step in the investigation of the unique solvability of the homotopy Dirichlet problem for p-harmonic maps into a geodesically complete manifold N of non-positive curvature.

Problem A. *Let (M,g) be a compact, m-dimensional Riemannian manifold with smooth boundary $\partial M \neq \emptyset$ and let N be a complete, possibly compact, n-dimensional Riemannian manifold without boundary. Assume also that N has non-positive sectional curvature or, more generally, that the universal covering of N supports a strictly convex exhaustion function. For any $p \geq 2$ and any given $f \in C^0\left(M,N\right)$, consider the p-Dirichlet problem*

$$\begin{cases} \Delta_p u = 0 & \text{on } M \\ u = f & \text{on } \partial M. \end{cases}$$

Has this p-Dirichlet problem a (unique) solution $u \in C^{1,\alpha}(\mathrm{int}(M), N) \cap C^0(M, N)$ *in the homotopy class of* f *relative to* ∂M*?*

Actually one expects Problem A to have a positive answer. A first evidence in this direction is given by the classical harmonic case. When $p = 2$, the Dirichlet problem for harmonic maps into non-positively curved manifolds has been solved by R. Schoen and S.T. Yau, who extended to non-compact targets a previous result due to R. Hamilton; see [Ham] and Theorem 8.5 in Chapter IX of [SY2]. Schoen and Yau's proof makes use of Hamilton's heat flow for harmonic maps. In view of the achievements in the papers quoted above, p-harmonic heat flow techniques look promising in obtaining a complete solution of Problem A even in case $p \neq 2$. However, so far and to the best of our knowledge, no significant progress in this direction has been made yet.
In the p-harmonic realm, a partial result is due to S.W. Wei [We2] who considered, for a compact target N and boundary datum $f \in C^0(M, N) \cap Lip(\partial M, N)$, a weaker version of Problem A. More precisely, in Theorem 7.1 of [We2], using a procedure similar to that introduced by F. Burstall in [Bu], and which is based on the conjugacy class of homomorphisms induced by the Sobolev maps on the fundamental groups, Wei discussed the solvability of the Dirichlet problem in the free homotopy class of the initial datum f. However, the conjugacy class of the induced homomorphism is not enough to determine completely the relative homotopy type of a given map. An easy counterexample can be constructed by considering the 2-dimensional torus $N = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and the compact manifold with boundary $M \subset \mathbb{T}^2$ given by $M = \{[(x_1, x_2)] \in \mathbb{T}^2 : 0 \leq x_1 \leq 1/2\}$. Then, choosing $f : M \to N$ defined by $f([(x_1, x_2)]) = [(3x_1, x_2)]$, one has that the inclusion map $i : M \hookrightarrow N$ is a harmonic map with $i|_{\partial M} = f|_{\partial M}$, which induces a homomorphism $i_\sharp$ conjugated to $f_\sharp$, but f and i are not homotopic relative to ∂M.
Nevertheless, since in [Wh] the relative $[p-1]$-homotopy type of a $W^{1,p}$ map is defined, one can minimize the p-energy among maps preserving such a relative $[p-1]$-homotopy type. The subtle point now is to show that the regularity theory of [HL] applies also in this case. This is one of the main points of the present paper that enables us to work out the program for non-positively curved targets started by B. White long ago. In this respect it is worth to mention the remarkable work by F. Duzaar and M. Fuchs, [DF2], where the authors prove the existence of a p-harmonic representative in the homotopy class of a map from a compact manifold without boundary into a compact target of non-negative curvature. Due to some non-trivial difficulties in the application of the regularity theory, the authors decided to take a different, and less direct, approach which is based on a perturbation technique. It is likely that their approach, when combined with the regularity theory of constrained maps by M. Fuchs, [Fu, Fu2], can be adapted to solve also the relative Dirichlet problem. However, the adaptations do not seem completely obvious.

Summarizing, the first main purpose of the paper is to give a direct proof of the following

Theorem B. *Let* M *be a compact manifold with boundary* $\partial M \neq \emptyset$*, and* N *be a compact manifold whose universal covering supports a strictly convex exhaustion function. Let* $f \in C^0(M, N) \cap Lip(\partial M, N)$*. Then, for any* $p \geq 2$*, there exists a* p*-harmonic map* $u \in C^{1,\alpha}(\mathrm{int}(M), N) \cap C^0(M, N)$ *which minimizes the* p*-energy among all the* $W^{1,p}$ *maps in the homotopy class of* f *relative to* ∂M*. In particular* u *is the unique solution of Problem A when* ${}^N\mathrm{Sect} \leq 0$*.*

We point out that the uniqueness part of the theorem is a consequence of [We2]. However, at the end of the paper, and in view of future developments in the non-compact case, [PV2], we shall include a very general uniqueness result that works for homotopic (rel ∂M) p-harmonic maps into a complete manifold of non-positive curvature. This is the second main result of the paper which represents a new contribution to the difficult comparison theory for p-harmonic maps in the presence of topology: in fact, it extends to domains with boundary the comparison result obtained in [V].

Theorem C. *Let (M,g) be a compact, m-dimensional Riemannian manifold with smooth boundary $\partial M \neq \emptyset$ and let N be a complete manifold such that ${}^N\mathrm{Sect} \leq 0$. Let $f \in C^0(M,N)$. Then, for any $p \geq 2$, there is at most a C^1 solution to the homotopy p-Dirichlet Problem A with datum f.*

We observe explicitly that our comparison requires that the p-harmonic maps at hand are continuous, even in case their images are confined in a regular ball of the target. It is an interesting open question whether, in this situation, the result extends to weakly p-harmonic maps. In the very recent [FR3], A. Fardoun and R. Regbaoui obtained a positive answer provided the regular ball has sufficiently small radius, but it is very reasonable that this constraint is technical rather than substantial and, hence, should be removed.

2. Existence of solutions

The proof of Theorem B, that will be presented in Subsection 2.5, requires some preliminary results of very different nature that are collected in the next three subsections.

2.1. Lipschitz approximation in relative homotopy class.

According to Section 4 in [Wh] a boundary datum $f \in C^0(M,N) \cap Lip(\partial M, N)$ has a $C^0(M,N) \cap W^{1,p}(M,N)$ representative in its homotopy class relative to ∂M. This fact will be extensively used in the proof of Theorem B. Actually, the representative can be chosen to be $Lip(M,N)$; see Proposition 2.3 below. This follows by combining the standard Whitney approximation result with the next Lemma which is implicitly contained in [Wh], see especially the proof of Theorem 4.1 there.

Lemma 2.1. *Let (M,g) be a compact m-dimensional Riemannian manifold with boundary $\partial M \neq \emptyset$. Then, there exists a Lipschitz map $u : M \to M$ satisfying the following conditions:*

- (a) *u is homotopic to the identity map id_M relative to ∂M.*
- (b) *u is a smooth retraction of a collar neighborhood V of ∂M onto ∂M. Moreover, the neighborhood V can be chosen as small as desired.*
- (c) *u is a diffeomorphism of $M \setminus V$ onto M.*

Proof. Fix an open collar neighborhood W of ∂M and let $\alpha : \partial M \times [0,+\infty) \to W$ be a diffeomorphism satisfying $\alpha(x,0) = x$. Note that, if we set $\alpha^{-1}(x) = (\alpha_1(x), \alpha_2(x))$, the map

$$r(x) = \alpha(\alpha_1(x), 0) : W \to \partial M$$

is a natural smooth retraction of W onto ∂M. Now, for any $0 \leq s < t$, let

$$\begin{aligned} \mathcal{M}_s^t &= \alpha(\partial M \times [s,t]) \\ \mathcal{M}_t &= \overline{M \backslash \mathcal{M}_0^t} \\ \mathcal{B}_t &= \alpha(\partial M \times t) \end{aligned}$$

so that

$$\mathcal{M}_t \cap \mathcal{M}_s^t = \mathcal{B}_t.$$

Observe that $\mathcal{M}_0 = M$ can be realized as the obvious gluing

$$M = \mathcal{M}_2 \cup_{\mathrm{id}_{\mathcal{B}_2}} \mathcal{M}_0^2$$

whereas

$$\mathcal{M}_1 = \mathcal{M}_2 \cup_{\mathrm{id}_{\mathcal{B}_2}} \mathcal{M}_1^2.$$

On the other hand, $\mathcal{M}_1^2$ is diffeomorphic to $\mathcal{M}_0^2$ via

$$\beta\left(x\right) = \alpha\left(\alpha_1\left(x\right), 2\alpha_2\left(x\right) - 2\right)$$

which keeps $\mathcal{B}_2$ fixed, i.e., $\beta = \mathrm{id}_{\mathcal{B}_2}$ on $\mathcal{B}_2$. It follows that the homeomorphism $\gamma : \mathcal{M}_1 \to M$ such that

$$\gamma\left(x\right) = \begin{cases} \mathrm{id}_{\mathcal{M}_2}\left(x\right), & x \in \mathcal{M}_2 \\ \beta\left(x\right), & x \in \mathcal{M}_1^2 \end{cases} \tag{3}$$

smooths out along the closed submanifold $\mathcal{B}_2 = \mathcal{M}_2 \cap \mathcal{M}_1^2$ and gives rise to a global diffeomorphism $\Gamma : \mathcal{M}_1 \to M$ satisfying $\Gamma = \gamma$ outside a small neighborhood of $\mathcal{B}_2$; see e.g. Theorem 1.9 of [Hi]. In particular,

$$\Gamma\left(x\right) = r\left(x\right), \text{ on } \mathcal{B}_1.$$

To conclude, we put

$$V = \alpha\left(\partial M \times [0,1)\right) \subset W$$

and define $u : M \to M$ by setting

$$u\left(x\right) = \begin{cases} r\left(x\right), & x \in \mathcal{M}_0^1 = \bar{V} \\ \Gamma\left(x\right), & x \in \mathcal{M}_1. \end{cases}$$

□

Remark 2.2. The same proof works if M is a non-compact manifold with compact boundary ∂M. Clearly, in this case, u is only $Lip_{loc}(M, N)$. In fact, note that the assumption that ∂M is compact is just used to smoothing out the homeomorphism γ along the submanifold B_2 and this is needed to obtain u satisfying the further condition (c) in the statement of Lemma. If we are not interested in smooth regularity, we can use directly γ, whose construction does not require any compactness assumption on ∂M. In this case, condition (c) has to be replaced by

(c)' u is a $BiLip_{loc}$-homeomorphism of $M \setminus V$ onto M.

Proposition 2.3. *Keeping the notation and assumptions of the previous Lemma, suppose we are given a map $f \in C^0\left(M, N\right) \cap Lip\left(\partial M, N\right)$, where (N, h) is an n-dimensional Riemannian manifold without boundary. Then, there exists $F \in Lip\left(M, N\right)$ which is homotopic to f relative to ∂M.*

Proof. Let $u : M \to M$ be the map defined in Lemma 2.1 and consider $\overline{f} = f \circ u : M \to M$. Then, $\overline{f} \in C^0\left(M\right) \cap Lip_{loc}\left(V\right)$ where V is a collar neighborhood of ∂M which retracts to ∂M via u. In particular $f = \overline{f}$ on ∂M. Let W be a smaller collar neighborhood of ∂M such that $\overline{W} \subset V$. Then, we can apply the standard approximation procedure by H. Whitney keeping $\overline{W}$ fixed (see e.g. [Le]) and obtain a Lipschitz map $F : M \to N$ with the desired properties. More precisely: (i) F is smooth on $M \backslash V$; (ii) $F = \overline{f}$ on $\overline{W}$ and (iii) F is homotopic to $\overline{f}$, relative to ∂M. □

Remark 2.4. Using the version of Lemma 2.1 observed in Remark 2.2, we can skip the assumption that M is compact and obtain that any $f \in C^0(M,N) \cap Lip_{loc}(\partial M, N)$ has a representative $F \in Lip_{loc}(M,N)$ in its homotopy class relative to ∂M.

2.2. **p-Minimizing tangent maps.** Another key ingredient in the proof of Theorem B is the fact that a manifold does not contain any p-minimizing tangent sphere provided its universal covering supports a smooth strictly convex function. This is the content of Proposition 2.6 that will be vital to apply the full regularity theory by Hardt-Lin. The conclusion for $p = 2$ was initially obtained in [SU1]. Nakauchi pointed out how to extend Schoen and Uhlenbeck's iteration process to $p > 2$, [N]. The general result for $p > 2$ and target supporting a strictly convex function was also observed in [WY].

Due to its importance, we shall present a rather detailed treatment.

First, we introduce the Sobolev spaces of maps that will be used throughout the paper. According to the Nash embedding theorem, we can assume that there is an isometric embedding $i : N \hookrightarrow \mathbb{R}^q$ of N into some Euclidean space. For all maps $u : M \to N$ we define $\breve{u} := i \circ u : M \to \mathbb{R}^q$. For $p > 1$, we denote by $W^{1,p}_{loc}(M,\mathbb{R}^q)$ (resp. $W^{1,p}(M,\mathbb{R}^q)$) the Sobolev space of maps $v : M \to \mathbb{R}^q$ whose component functions and their first weak derivatives are in $L^p_{loc}(M)$ (resp. in $L^p(M)$). Moreover we define

$$W^{1,p}_{loc}(M,N) := \{v \in W^{1,p}_{loc}(M,\mathbb{R}^q) : v(x) \in N \text{ for a.e. } x \in M\},$$
$$W^{1,p}(M,N) := \{v \in W^{1,p}(M,\mathbb{R}^q) : v(x) \in N \text{ for a.e. } x \in M\}.$$

Finally we will say that $v \in W^{1,p}(M,N)$ has boundary trace f if $\breve{v} - \breve{f} \in W^{1,p}_0(M,\mathbb{R}^q)$, where $W^{1,p}_0(M,\mathbb{R}^q)$ denotes the closure of $C^\infty_c(M,\mathbb{R}^q)$ in the $W^{1,p}(M,\mathbb{R}^q)$ norm.

Following [HL, p. 572] we say that a map $\bar{\psi} \in W^{1,p}_{loc}(\mathbb{R}^{l+1},N)$ is a p-minimizing tangent map (p-MTM) from $\mathbb{R}^{l+1}$ to N if $\bar{\psi}$ minimizes the p-energy on compact sets and $\bar{\psi}$ is homogeneous of degree 0, that is, $\partial\bar{\psi}/\partial r = 0$ a.e., r being the radial coordinate. Clearly, here we are thinking of $\bar{\psi}$ as an $\mathbb{R}^q$-valued map once N is isometrically embedded in the Euclidean space $\mathbb{R}^q$. Note that $\bar{\psi} \in W^{1,p}_{loc}(\mathbb{R}^{l+1},N)$ is homogeneous of degree 0 if and only if there exists $\psi \in W^{1,p}(\mathbb{S}^l,N)$ such that

$$\bar{\psi}(x) := \psi\left(\frac{x}{|x|}\right), \quad \forall x \neq 0. \tag{4}$$

Indeed, condition (4) clearly implies that $\partial\bar{\psi}/\partial r = 0$ a.e. On the other hand, if $\bar{\psi} \in W^{1,p}_{loc}(\mathbb{R}^{l+1},N)$ is homogeneous of degree 0, since by Fubini's theorem $\bar{\psi}(\cdot,\theta) \in W^{1,p}_{loc}(\mathbb{R}_{>0},N)$ for a.e. $\theta \in \mathbb{S}^l$, we deduce that $\bar{\psi}(\cdot,\theta)$ is constant a.e. Therefore, $\psi(\theta) = \bar{\psi}(\cdot,\theta)$ satisfies (4) and, again by Fubini, it is $W^{1,p}(\mathbb{S}^l,N)$.

Lemma 2.5. *Assume that $\bar{\psi} \in W^{1,p}_{loc}(\mathbb{R}^{l+1},N)$ satisfies (4) for some $\psi \in W^{1,p}(\mathbb{S}^l,N)$. Then $\bar{\psi} : \mathbb{R}^{l+1} \to N$ is weakly p-harmonic (in the sense of (2)) if and only if $\psi : \mathbb{S}^l \to N$ is weakly p-harmonic.*

Proof. Let $(r,\theta) \in \mathbb{R}_{>0} \times \mathbb{S}^l$ be local polar coordinates on $\mathbb{R}^{l+1}$. Namely, we suppose to have chosen local angular coordinates $\{\theta^1,...,\theta^l\}$ on $\mathbb{S}^l$ so that $\{r,\theta^1,...,\theta^l\}$ is a local coordinates system for $\mathbb{R}^{l+1} \setminus \{0\}$.

Having fixed an isometric embedding $i : N \to \mathbb{R}^q$ with second fundamental form $\mathcal{A}$, we let $\check{\bar{\psi}} = i \circ \bar{\psi} : \mathbb{R}^{l+1} \to \mathbb{R}^q$ and $\check{\psi} = i \circ \psi : \mathbb{S}^l \to \mathbb{R}^q$. By assumption $\bar{\psi}(r, \theta) = \psi(\theta)$, where $\psi \in W^{1,p}(\mathbb{S}^l, N)$.

Let $\bar{\varphi} \in C^\infty_c(\mathbb{R}^{l+1}, \mathbb{R}^q)$. Then, we have

$$(D\check{\bar{\psi}} \cdot D\bar{\varphi})(r, \theta) = r^{-2}(D\check{\psi} \cdot D(\bar{\varphi}(r, \cdot)))(\theta),$$

and

$$|D\check{\bar{\psi}}|^2(r, \theta) = r^{-2}|D\check{\psi}|^2(\theta).$$

Whence, it follows that, for every $R > 0$,

$$E_p(\bar{\psi}|_{\mathbb{B}_R(0)}) = E_p(\psi) \int_0^R r^{l-p} dr \tag{5}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^{l+1}} |D\check{\bar{\psi}}|^{p-2} \left\{ D\check{\bar{\psi}} \cdot D\bar{\varphi} + \mathcal{A}(D\check{\bar{\psi}}, D\check{\bar{\psi}}) \cdot \bar{\varphi} \right\} dx \\ &= \int_0^\infty r^{l-p} \int_{\mathbb{S}^l} |D\check{\psi}|^{p-2}(\theta) \left\{ D\check{\psi} \cdot D(\bar{\varphi}(r, \cdot)(\theta) \right. \\ &\left. + \mathcal{A}(D\check{\psi}, D\check{\psi}) \cdot \bar{\varphi}(r, \theta) \right\} d\sigma(\theta) dr. \end{aligned} \tag{6}$$

If $p \geq l + 1$, from (5) we must conclude that $E_p(\psi)$=0 and, therefore, that ψ and $\bar{\psi}$ are constant. In particular, they are both trivially p-harmonic.

Suppose that $p < l + 1$. Since, for each $r > 0$, $\bar{\varphi}(r, \cdot) \in C^\infty(\mathbb{S}^l, \mathbb{R}^q)$, recalling the extrinsic definition of (weak) p-harmonicity given in (2), from (6) we deduce that if $\psi : \mathbb{S}^l \to N$ is weakly p-harmonic, then $\bar{\psi} : \mathbb{R}^{l+1} \to N$ is weakly p-harmonic. On the other hand, assume that $\bar{\varphi}$ has the form $\bar{\varphi}(r, \theta) = \varphi(\theta)\nu(r)$, where $\varphi \in C^\infty(\mathbb{S}^l, \mathbb{R}^q)$ and $\nu \in C^\infty_c([0, \infty))$ is such that $\nu^{2k+1}(0) = 0$ for all $k \geq 0$. Then $\bar{\varphi} \in C^\infty_c(\mathbb{R}^{l+1}, \mathbb{R}^q)$ and (6) becomes

$$\begin{aligned} &\int_{\mathbb{R}^{l+1}} |D\check{\bar{\psi}}|^{p-2} \left\{ D\check{\bar{\psi}} \cdot D\bar{\varphi} + \mathcal{A}(D\check{\bar{\psi}}, D\check{\bar{\psi}}) \cdot \bar{\varphi} \right\} dx \\ &= \left\{ \int_0^\infty r^{l-p} \nu(r) dr \right\} \left\{ \int_{\mathbb{S}^l} |D\check{\psi}|^{p-2} \left[D\check{\psi} \cdot D\varphi + \mathcal{A}(D\check{\psi}, D\check{\psi}) \cdot \varphi \right] d\sigma \right\}, \end{aligned}$$

proving that $\psi : \mathbb{S}^l \to N$ is weakly p-harmonic whenever $\bar{\psi} : \mathbb{R}^{l+1} \to N$ is weakly p-harmonic. □

Proposition 2.6. *Suppose that N is compact and that $\xi \in W^{1,p}_{loc}(\mathbb{R}^l, N)$ is a p-MTM. If $l \leq [p]$ then ξ is constant and, if $l > [p]$ and N does not support any non-constant p-MTM from $\mathbb{R}^j$ into N, $j = 1, ..., l - 1$, then ξ has at most an isolated singularity at the origin. In particular, $\xi|_{\mathbb{S}^{l-1}} \in C^{1,\alpha}(\mathbb{S}^{l-1}, N)$. Moreover, if the universal cover $\tilde{N}$ of N supports a strictly convex function, then every p-MTM from $\mathbb{R}^l$ to N is constant, for every $l \geq 1$.*

Proof. Indeed, the case $l \leq [p]$ follows directly from Theorem 4.5 of [HL]. For the general case we proceed by induction. Let $l > [p]$ and suppose that every p-MTM from $\mathbb{R}^j$ to N is trivial for every $j = 1, \dots, l - 1$. Let $\xi : \mathbb{R}^l \to N$ be a p-MTM. Then, by Theorem 4.5 in [HL] the set of singular points of ξ is discrete (possibly empty) and by homogeneity it reduces to the sole origin. In particular, by Corollary 2.6 and Theorem 3.1 in [HL], we deduce that $\xi|_{\mathbb{S}^{l-1}} : \mathbb{S}^{l-1} \to N$ is $C^{1,\alpha}$ and it is p-harmonic thanks to Lemma 2.5. To conclude, in case $\tilde{N}$ supports a convex function, we can apply Theorem 1.4 in [WY] and obtain that ξ is constant. □

2.3. **Extending relative d-homotopies.** Fix a triangulation of ∂M and extend it to a triangulation of M. Thus M is a CW-complex and ∂M is a subcomplex of M, see [Whd, Mu]. Let M^d denote that d-skeleton of M.

Two continuous maps $v, f : M \to N$ are said to be d-homotopic relative to $M^d \cap \partial M$ (or, equivalently, they have the same d-homotopy type) if there exists a continuous map $H^d : [0,1] \times M^d \to N$ such that $H^d(0,x) = v(x)$, $H^d(1,x) = f(x)$ for all $x \in M^d$ and $H^d(\cdot,x) = f(x) = v(x)$ for all $x \in M^d \cap \partial M$. Clearly, when $d \geq \dim M$ the relative d-homotopy type of maps is nothing but the usual homotopy type relative to ∂M.

By the homotopy extension property of the couple (M, M^d) we already know that if v and f have the same d-homotopy type, then H^d extends to a full homotopy $H : M \to N$ such that $H(0,x) = v(x)$. In this subsection, under the assumption that the target manifold N is aspherical, we construct a special extension H of H^d satisfying the further requirements $H(1,x) = f(x)$ for every $x \in M$ and $H(\cdot,x) = f(x) = v(x)$ for every $x \in \partial M$.

Recall that N is said to be aspherical if each homotopy group $\pi_k(N)$ of N is trivial for $k \geq 2$.

Proposition 2.7. *Let $v, f \in C^0(M,N)$ and assume that N is aspherical. If v and f have the same relative d-homotopy type, $d \geq 1$, then they have the same relative homotopy type.*

Proof. By assumption, we know that there exists a continuous map $H^d : [0,1] \times M^d \to N$ such that $H^d(0,x) = v(x)$, $H^d(1,x) = f(x)$ for all $x \in M^d$ and $H^d(\cdot,x) = f(x) = v(x)$ for all $x \in M^d \cap \partial M$.

Using the aspherical structure of N in a classical manner (see e.g [Hat]), we are going to show that H^d extends to a homotopy H^{d+1} between v and f on M^{d+1} relative to $M^{d+1} \cap \partial M$, i.e., $H^{d+1} : [0,1] \times M^{d+1} \to N$ is a continuous function such that $H^{d+1}(0,\cdot) = v(\cdot)$, $H^{d+1}(1,\cdot) = f(\cdot)$ and $H^{d+1}(\cdot,x) = f(x) = v(x)$ for all $x \in M^{d+1} \cap \partial M$. Clearly we can assume $1 \leq d < m$ for otherwise there is nothing to prove.

The $(d+1)$-skeleton M^{d+1} is obtained as $M^{d+1} = M^d \cup \left(\cup_\alpha e_\alpha^{d+1}\right)$, where e_α^{d+1} are open $(d+1)$-cells with attaching maps $\psi_\alpha : \mathbb{S}^d \to M^d$ and corresponding characteristic maps $\bar{\psi}_\alpha : \mathbb{D}^{d+1} \to M^{d+1}$. Note that $[0,1] \times M^d \subseteq ([0,1] \times M)^{d+1}$ and H^d extends to a continuous function $H^{d+1} : ([0,1] \times M)^{d+1} \to N$ by setting $H^{d+1}(0,x) = v(x)$ and $H^{d+1}(1,x) = f(x)$ on $\{0,1\} \times \bar{e}_\alpha^{d+1}$.

The CW complex $[0,1] \times M^{d+1} \subseteq ([0,1] \times M)^{d+2}$ is obtained by attaching to $([0,1] \times M)^{d+1}$ the open cells $E_\alpha^{d+2} = (0,1) \times e_\alpha^{d+1}$ via $\Psi_\alpha : \mathbb{S}^{d+1} \to ([0,1] \times M)^{d+1}$ where $\mathbb{S}^{d+1} \approx ([0,1] \times \mathbb{S}^d) \cup (\{0,1\} \times \mathbb{D}^{d+1})$, $\Psi_\alpha = \mathrm{id} \times \psi_\alpha$ on $[0,1] \times \mathbb{S}^d$ and $\Psi_\alpha = \{0,1\} \times \bar{\psi}_\alpha$ on $\{0,1\} \times \mathbb{D}^{d+1}$. Let us show how to define H^{d+1} on $([0,1] \times M)^{d+1} \cup E_\alpha^{d+2}$. Since ∂M is a subcomplex of M, we have that either $e_\alpha^{d+1} \subset \partial M$ or $e_\alpha^{d+1} \subset \mathrm{int}(M)$. In the first case we just define, for all $(t,x) \in E_\alpha^{d+2}$, $H^{d+1}(t,x) = f(x) = v(x)$. In the second case, note that H^{d+1} is already defined on $\Psi_\alpha(\mathbb{S}^{d+1})$. Since $d+1 \geq 2$ and N is aspherical, the composition $H^{d+1} \circ \Psi_\alpha : \mathbb{S}^{d+1} \to N$ is null-homotopic and, therefore, it extends to a continuous map $H^{d+1} \circ \Psi_\alpha : E_\alpha^{d+2} \to N$. This implies that H^{d+1} itself extends to a continuous map from $([0,1] \times M)^{d+1} \cup E_\alpha^{d+2}$ into N.

Now, repeating the same procedure inductively for all E_α^{d+2} and for all the k-skeletons M^k, $d < k \leq m$, we complete the construction of the desired relative homotopy H.

□

2.4. **The d-homotopy type of $W^{1,p}$ maps.** Let d be the greatest integer less than or equal to $p-1$ and let M^d be a d-dimensional skeleton of M. Clearly, here we mean $M^d \equiv M$ for $p-1 \geq m$. Recall from the previous subsection that the d homotopy type of a continuous map from M to N is the homotopy type of its restriction to M^d. According to the work of White [Wh], each $u \in W^{1,p}(M,N)$ with boundary trace f has a d-homotopy type $u_\sharp[M^d(\mathrm{rel}\,\partial M)]$. This d-homotopy type is a homotopy class (relative to ∂M) of continuous mappings from M^d into N such that:

(1) If $\{u_i\} \subset W^{1,p}(M,N)$ have boundary trace h, $\|\check{u}_i - \check{u}\|_p \to 0$, and $\|du_i\|_p$ is uniformly bounded, then
$$(u_i)_\sharp[M^d(\mathrm{rel}\,\partial M)] = u_\sharp[M^d(\mathrm{rel}\,\partial M)]$$
for sufficiently large i.

(2) If $u \in W^{1,p}(M,N)$ has boundary trace f and is continuous at each $x \in M^d$, then
$$u_\sharp[M^d(\mathrm{rel}\,\partial M)] = [(u|_{M^d})(\mathrm{rel}\,\partial M)].$$

(3) The set
$$\{u_\sharp[M^d(\mathrm{rel}\,\partial M)] : u \in W^{1,p}(M,N) \text{ has boundary trace } f\}$$
is equal to
$$\{[(\varphi|_{M^d})(\mathrm{rel}\,\partial M)] : \varphi \in C^0(M^{d+1},N),$$
$$\varphi(x) = f(x) \text{ for } x \in M^d \cap \partial M\}.$$

The purpose of this subsection is to point out the following property of the d-homotopy type whose application will be basic in the proof of Theorem B to apply the regularity theory of [HL].

Theorem 2.8. *Let M be a compact m-dimensional manifold with (possibly empty) boundary ∂M and let N be a compact manifold without boundary. Let $f \in Lip(\partial M, N)$ and let $v \in W^{1,p}(M,N)$ be a map with boundary trace f. For every $x \in M$ there exists an open set $x \in \Omega_x \subset M$ (independent of v) with smooth boundary $\partial\Omega_x$, which satisfies the following property: for any other map $w \in W^{1,p}(M,N)$ such that $w|_{\partial M} = v|_{\partial M}$ in the trace sense and $w \equiv v$ on $M \setminus \Omega_x$ it holds*

$$w_\sharp[M^d(\mathrm{rel}\,\partial M)] = v_\sharp[M^d(\mathrm{rel}\,\partial M)]. \tag{7}$$

Proof. Let us assume that $p-1 < m$. Otherwise, the d-skeleton is the entire manifold M and the Sobolev $W^{1,p}$-maps are continuous, therefore the conclusion is trivial. We consider three cases.

First suppose that $x \in \mathrm{int}(M)$ and that $x \notin M^d$. Since M^d is closed in M we can choose the open set Ω_x such that $x \in \Omega_x \subset\subset \mathrm{int}(M)\setminus M^d$ and $\partial\Omega_x$ is smooth. By the construction of the d-homotopy type given in Section 3 of [Wh], and up to choosing $\delta > 0$ small enough in (the version for manifolds with boundary of) Proposition 3.2 therein, it is clear that in this case perturbing a $W^{1,p}$ map in Ω_x does not affect the d-homotopy type of the map.

Now, let $x \in M^d \cap \mathrm{int}(M)$. Consider, for $\epsilon > 0$ small enough, a normal closed geodesic ball $\bar{B}_\epsilon(x)$ centered at x. Choose a triangulation T_x of $\bar{B}_\epsilon(x)$ such that x is not contained in the d-skeleton T_x^d of T_x (to this purpose, one can for instance take such a construction on the Euclidean unit closed ball, and make use of the diffeomorphism with $\bar{B}_\epsilon(x)$ given by the normal coordinates). Note that T_x induces a triangulation of $\partial\bar{B}_\epsilon(x)$. Choosing a triangulation of ∂M gives, together with $T_x|_{\partial \bar{B}_\epsilon(x)}$, a triangulation of $\partial M \cup \partial \bar{B}_\epsilon(x) = \partial(M \setminus \bar{B}_\epsilon(x))$. A classical result ensures us that this triangulation of the boundary can be extended to a triangulation of all of $M \setminus \bar{B}_\epsilon(x)$ [Whd, Mu]. This latter, together with T_x, forms a new triangulation of M whose d-skeleton $(M^d)'$ does not contain x. According to the previous case there exists an open set with smooth boundary Ω_x such that, given maps v and w as in the statement, we have

$$w_\sharp[(M^d)'(\mathrm{rel}\,\partial M)] = v_\sharp[(M^d)'(\mathrm{rel}\,\partial M)].$$

To conclude this case, we recall that, thanks to Proposition 3.5 of [Wh], the d-homotopy type of a $W^{1,p}$ map does not depend on the choice of the triangulation.
Finally, suppose that $x \in \partial M$. Using the notation of Lemma 2.1, let $y \in \partial V$ be a point satisfying $u(y) = x$. Since $y \in \mathrm{int}(M)$, according to previous paragraphs there exists an open set Ω'_y, $y \in \Omega'_y \subset\subset \mathrm{int}(M)$, such that perturbing a $W^{1,p}$ map inside Ω'_y does not change the d-homotopy type of the map. Let $\Omega_x = u(\Omega'_y \cap (M \setminus V))$. Since $u|_{M\setminus V}$ is a differomorphism onto M, then Ω_x is an open set of M containing x. Suppose that v, w are two $W^{1,p}(M,N)$ maps as in the statement. Then, also $v \circ u$ and $w \circ u$ are maps in $W^{1,p}(M,N)$. Furthermore, by construction of u, it holds $v \circ u|_V = w \circ u|_V = f \circ u|_V$, where f is the trace value of v and w at the boundary. This in turn implies $v \circ u|_{\partial M} = w \circ u|_{\partial M}$ in the trace sense, and $v \circ u(z) = w \circ u(z)$ for each $z \in M \setminus \Omega'_y$, since either $u(z) \in \partial M$ or $u(z) \in M \setminus \Omega_x$. Hence,

$$(w \circ u)_\sharp[M^d(\mathrm{rel}\,\partial M)] = (v \circ u)_\sharp[M^d(\mathrm{rel}\,\partial M)].$$

On the other hand, by the construction of the relative d-homotopy type of $W^{1,p}$ maps given in [Wh] it is clear that

$$\begin{aligned}(v \circ u)_\sharp[M^d(\mathrm{rel}\,\partial M)] &= v_\sharp[M^d(\mathrm{rel}\,\partial M)],\\ (w \circ u)_\sharp[M^d(\mathrm{rel}\,\partial M)] &= w_\sharp[M^d(\mathrm{rel}\,\partial M)].\end{aligned}$$

This latter, in turn, implies (7) as aimed. To conclude, observe that since ∂M is smooth, up to possibly restrict the set Ω_x, we can require that Ω_x has smooth boundary. □

2.5. **Proof of Theorem B.** For the sake of clarity, we will divide the proof in four steps.

Step 1. Existence of a minimizer in the d-homotopy class of f. Define $\mathcal{H}^d_f$ as the space of maps $u \in W^{1,p}(M,N)$ such that $u|_{\partial M} = f|_{\partial M}$ in the trace sense and f and u have the same relative d-homotopy type, i.e.

$$\begin{aligned}\mathcal{H}^d_f :=\{u \in W^{1,p}(M,N) : \check{u} - \check{f} \in W^{1,p}_0(M,\mathbb{R}^q) \text{ and}\\ u_\sharp[M^d(\mathrm{rel}\,\partial M)] = f_\sharp[M^d(\mathrm{rel}\,\partial M)]\}.\end{aligned}$$

According to Proposition 2.3, there is no loss of generality if we assume that $f \in Lip(M,N)$. In particular $f \in \mathcal{H}^d_f$ and, therefore,

$$\mathcal{I}^d_f := \inf_{u \in \mathcal{H}^d_f} E_p(u) < +\infty.$$

Let $\{v_j\}_{j=1}^\infty \subset \mathcal{H}^d_f$ be a sequence minimizing the p-energy in $\mathcal{H}^d_f$, i.e. $E_p(v_j) \to \mathcal{I}^d_f$ as $j \to \infty$. For the ease of notation, throughout all the proof we will keep the same set of indexes each time we will extract a subsequence from a given sequence.

Since N is compact, then $\{\check{v}_j\}_{j=1}^{\infty}$ is bounded in $W^{1,p}(M,\mathbb{R}^q)$ and, up to choosing a subsequence, $\check{v}_j$ converges to some $\check{v} \in W^{1,p}(M,\mathbb{R}^q)$ weakly in $W^{1,p}$. Since M is compact, $\{\check{v}_j\}_{j=1}^{\infty}$ is bounded in $W^{1,p'}(M,\mathbb{R}^q)$ for every $p' \leq p$ which satisfies also $p' < m$. By the Kondrachov theorem, [Au] p.55, $\check{v}_j$ converges strongly in $L^s(M,\mathbb{R}^q)$ for any $1 < s < (mp')/(m-p')$, notably for $s = p$, and hence pointwise almost everywhere. Since N is properly embedded, this implies $\check{v}(x) \in N$ for a.e. $x \in M$, so that we can define $v \in W^{1,p}(M,N)$ by $v = \check{v}$. Since $\check{v}_j - \check{f} \in W_0^{1,p}(M,\mathbb{R}^q)$ for all j, the weak limit $\check{v} - \check{f} \in W_0^{1,p}(M,\mathbb{R}^q)$.
By the lower semicontinuity of E_p we have

$$E_p(v) \leq \liminf_{j\to\infty} E_p(v_j) = \mathcal{I}_f^d. \tag{8}$$

Since $\{\check{v}_j\}_{j=1}^{\infty}$ is bounded in $W^{1,p}(M,\mathbb{R}^q)$ and $\|\check{v}_j - \check{v}\|_p \to 0$ as $j \to \infty$, by the property (1) of the d-homotopy type of maps we deduce that

$$v_\sharp[M^d(\operatorname{rel}\partial M)] = (v_j)_\sharp[M^d(\operatorname{rel}\partial M)] = f_\sharp[M^d(\operatorname{rel}\partial M)]$$

for j large enough, which implies that $v \in \mathcal{H}_f^d$. It then follows from (8) that

$$\mathcal{I}_f^d \leq E_p(v) \leq \mathcal{I}_f^d,$$

so that $E_p(v) = \mathcal{I}_f^d$, i.e. v minimizes the energy in $\mathcal{H}_f^d$.

Step 2. Regularity of the minimizer. We show that the regularity theory of [HL] applies to v, as already remarked on page 3 of [Wh]. Clearly, the only interesting case is $d := [p] - 1 \leq m - 1$.

First of all, we note that, for every $x \in M$, there exists an open set with smooth boundary $\Omega_x \ni x$ such that $v|_{\Omega_x}$ is a minimizer for the p-energy among all the maps $w \in W^{1,p}(\Omega_x, N)$ which have the same trace boundary of v on $\partial\Omega_x$, that is $v|_{\partial\Omega_x} = w|_{\partial\Omega_x}$ in the trace sense. To see this, let Ω_x be the open set given by Theorem 2.8. We can extend w to $\bar{w} \in W^{1,p}(M,N)$ by setting $\bar{w} = v$ on $M \setminus \Omega_x$. An application of Theorem 2.8 gives that $\bar{w} \in \mathcal{H}_f^d$. Then, $E_p(v) \leq E_p(\bar{w})$. To conclude, we note that

$$E_p(v|_{B_\epsilon}) + E_p(v|_{M\setminus B_\epsilon}) = E_p(v) \leq E_p(\bar{w}) = E_p(w) + E_p(v|_{M\setminus B_\epsilon}).$$

This minimizing property enables us to apply the partial interior regularity and deduce that the singular set $\mathcal{S}(v)$ of v is empty if $p > m$ and it is a relatively closed subset of zero $(m-p)$-Hausdorff dimension if $p \leq m$. Moreover, v is $C^{1,\alpha}$ on $\operatorname{int}(M) \setminus \mathcal{S}(v)$; see Corollary 2.6 and Theorem 3.1 in [HL].

The full interior regularity is now obtained from Theorem 4.5 of [HL] because, according to Proposition 2.6 above, every p-minimizing tangent map $\xi : \mathbb{R}^{l+1} \to N$ is constant, for every $l \geq 1$.

Finally, we observe that the boundary regularity theory developed in Section 5 of [HL] works for a Lipschitz boundary datum f. Therefore we can conclude that the minimizer v is $C^{0,\alpha}$ on M.

Step 3. On the relative homotopy class of the minimizer. It remains to prove that the minimizer v is homotopic to the datum f relative to ∂M. To this end, recall that M is realized as a polyhedral complex, hence a CW complex, in such a way that ∂M is a subcomplex. By construction, we know that v has the same $d(\geq 1)$-homotopy type of f

relative to $M^d \cap \partial M$. Note also that N is aspherical. Indeed, since its universal covering $\tilde{N}$ supports a strictly convex exhaustion function, by standard Morse theory $\tilde{N}$ is diffeomorphic to $\mathbb{R}^n$. The desired conclusion now follows from a direct application of Proposition 2.7.

Step 4. Non-positively curved targets. Suppose now that the compact manifold N has non-positive sectional curvature so that, in particular, its universal covering $\tilde{N}$ is a Cartan-Hadamard manifold. By the Hessian comparison theorem, the square of the distance function on $\tilde{N}$ is a strictly convex exhaustion function. Therefore, by the preceding steps, the homotopy p-Dirichlet problem has a solution $v \in C^{1,\alpha}(\mathrm{int}(M)) \cap C^0(M)$. Applying Theorem 8.5 (1) of [We2] we conclude that such a solution is unique.

This completes the proof of the Theorem.

3. A general uniqueness result

In Theorem B, the uniqueness property enjoyed by solutions of the p-Dirichlet problem is obtained from a result by W. Wei. In this Section we extend Wei's result to solutions of the homotopic p-Dirichlet problem in case the target manifold is non-compact. The construction via the quotient manifold $\hat{N}$ proposed in the proof below comes back to Schoen and Yau [SY1], which studied the moduli space of harmonic maps when M is a complete non-compact manifold with finite volume. Subsequently, in [PRS] it was observed that it is enough for M to be parabolic, while a generalization of Schoen and Yau's uniqueness results to p-harmonic maps has been obtained in [V]. In particular, in [V] there were introduced the convexity result stated below as Lemma 3.1 and the "mixed" vector field X used here, which in turn inspires to [PRS] and [HPV].

Proof (of Theorem C). Suppose u and v are two $C^1(M,N)$ solutions to Problem A. Let $P_M : \tilde{M} \to M$ and $P_N : \tilde{N} \to N$ be the universal Riemannian covers of M and N, respectively. Note that $\tilde{M}$ is a simply connected manifold with non-empty boundary $\partial\tilde{M}$ (which is in general neither compact nor simply connected) such that $P_M(\partial\tilde{M}) \equiv \partial M$. The fundamental groups $\pi_1(M,*)$ and $\pi_1(N,*)$ act as groups of isometries on $\tilde{M}$ and $\tilde{N}$ respectively, so that $M = \tilde{M}/\pi_1(M,*)$ and $N = \tilde{N}/\pi_1(N,*)$. Let $\mathrm{dist}_{\tilde{N}} : \tilde{N} \times \tilde{N} \to \mathbb{R}$ be the distance function on $\tilde{N}$. Since ${}^{\tilde{N}}\mathrm{Sect} \le 0$, we know that $\mathrm{dist}_{\tilde{N}}$ is smooth on $(\tilde{N} \times \tilde{N}) \setminus \tilde{D}$, where $\tilde{D}$ is the diagonal set $\{(\tilde{x},\tilde{x}) : \tilde{x} \in \tilde{N}\}$, and $\mathrm{dist}^2_{\tilde{N}}$ is smooth on $\tilde{N} \times \tilde{N}$. Now $\pi_1(N,*)$ acts on $\tilde{N} \times \tilde{N}$ as a group of isometries by

$$\beta(\tilde{x},\tilde{y}) = (\beta(\tilde{x}),\beta(\tilde{y})) \qquad \text{for } \beta \in \pi_1(N,*).$$

Thus $\mathrm{dist}^2_{\tilde{N}}$ induces a smooth function

$$\hat{r}^2 : \hat{N} \to \mathbb{R},$$

where we have defined

$$\hat{N} := (\tilde{N} \times \tilde{N})/\pi_1(N,*).$$

Let $U : M \times [0,1] \to N$ be a continuous relative homotopy between u and v. Since $\tilde{M}$, hence $\tilde{M} \times [0,1]$, is simply connected, U lifts to a homotopy $\tilde{U}$ between $\tilde{u}(\cdot) := \tilde{U}(\cdot,0)$ and $\tilde{v}(\cdot) := \tilde{U}(\cdot,1)$ relative to $\partial\tilde{M}$. Clearly, $P_N(\tilde{u}) = u(P_M)$ and $P_N(\tilde{v}) = v(P_M)$. Since Riemannian coverings are local isometries, $\tilde{u}$ and $\tilde{v}$ are p-harmonic maps and

$$|d\tilde{u}|(\tilde{q}) = |du|(P_M(\tilde{q})), \qquad |d\tilde{v}|(\tilde{q}) = |dv|(P_M(\tilde{q})).$$

Now, $\pi_1(M,*)$ acts as a group of isometries on $\tilde{M}$ and we have

$$\tilde{u}(\gamma(\tilde{q})) = u_\sharp(\gamma)\tilde{u}(\tilde{q}), \qquad \tilde{v}(\gamma(\tilde{q})) = v_\sharp(\gamma)\tilde{v}(\tilde{q}), \qquad \forall \tilde{q} \in \tilde{M}, \gamma \in \pi_1(M,*), \tag{9}$$

where $u_\sharp, v_\sharp : \pi_1(M,*) \to \pi_1(N,*)$ are the induced homomorphism and $u_\sharp \equiv v_\sharp$ since u is homotopic to v.
Thus, the map $\tilde{j} : \tilde{M} \to \tilde{N} \times \tilde{N}$ defined by $\tilde{j}(\tilde{x}) := (\tilde{u}(\tilde{x}), \tilde{v}(\tilde{x}))$ induces via (9) a map

$$j : M \to \hat{N}.$$

Furthermore, we can construct a vector valued 1-form $J \in T^*M \otimes j^{-1}T\hat{N}$ along j by projecting via (9) the vector valued 1-form $\tilde{J}$ along $\tilde{j}$ defined as

$$\tilde{J} := \left(\mathcal{K}_p\left(\tilde{u}\right), \mathcal{K}_p\left(\tilde{v}\right)\right) \quad \in T^*\tilde{M} \otimes \tilde{j}^{-1}T\left(\tilde{N} \times \tilde{N}\right).$$

Here and on, the symbol $\mathcal{K}_p\left(\tilde{u}\right)$ stands for

$$\mathcal{K}_p\left(\tilde{u}\right) := |d\tilde{u}|^{p-2}d\tilde{u}.$$

Consider the vector field on M given by

$$X|_q := \left[d\hat{r}^2|_{j(q)} \circ J|_q\right]^\sharp.$$

Note that

$$X|_q := dP_M|_{\tilde{q}} \circ \tilde{X}\Big|_{\tilde{q}}, \tag{10}$$

where

$$\tilde{X}\Big|_{\tilde{q}} := \left[d\left(\mathrm{dist}^2_{\tilde{N}}\right)\Big|_{\tilde{j}(\tilde{q})} \circ \tilde{J}\Big|_{\tilde{q}}\right]^\sharp.$$

We claim that (10) is well defined. To this end, let $S_{\tilde{q}} \in T_{\tilde{q}}\tilde{M}$ be an arbitrary vector and let $\tilde{q}' \in P_M^{-1}(q) \subset T\tilde{M}$. If $\tilde{q}' \neq \tilde{q}$, there exists $\gamma \in \pi_1(M,*)$ such that $q' = \gamma q$. Then,

$$\tilde{J}|_{\gamma\tilde{q}}(d\gamma\left(S_{\tilde{q}}\right)) = \left(d\left[u_\sharp(\gamma)\right]\left(\mathcal{K}_p\left(\tilde{u}\right)\left(S_{\tilde{q}}\right)\right), d\left[v_\sharp(\gamma)\right]\left(\mathcal{K}_p\left(\tilde{v}\right)\left(S_{\tilde{q}}\right)\right)\right).$$

Since u is homotopic to v, $u_\sharp = v_\sharp$. Moreover $\mathrm{dist}_{\tilde{N}}$ is equivariant with respect to the action of $\pi_1(N)$ on $\tilde{N} \times \tilde{N}$, i.e.

$$\mathrm{dist}_{\tilde{N}}(\beta\tilde{x}_1, \beta\tilde{x}_2) = \mathrm{dist}_{\tilde{N}}(\tilde{x}_1, \tilde{x}_2), \qquad \forall \beta \in \pi_1(N),\ x_1, x_2 \in \tilde{N}.$$

Then

$$dP_M|_{\tilde{q}} \circ \left[d\left(\mathrm{dist}^2_{\tilde{N}}\right)\Big|_{\tilde{j}(\tilde{q})} \circ \tilde{J}\Big|_{\tilde{q}}\right]^\sharp$$

does not depend on the choice of $\tilde{q} \in P_M^{-1}(q)$.
Now, we recall the following "convexity" result of [V].

Lemma 3.1. *For all $q \in M$ and for any choice of $\tilde{q} \in P_M^{-1}(q)$ we have*

$$\mathrm{tr}_{\tilde{M}}\, {}^{\tilde{N}\times\tilde{N}}\mathrm{Hess}\,\mathrm{dist}^2_{\tilde{N}}\,|_{\tilde{j}(\tilde{q})}\left(d\tilde{j}, \tilde{J}\right) \geq 0 \tag{11}$$

Moreover, having fixed an orthonormal frame $\tilde{E}_i$ in $T_{\tilde{q}}\tilde{M}$, with $i = 1, \dots, m$, the equality holds in (11) if and only if there are parallel vector fields Z_i, defined along the unique geodesic $\gamma_{\tilde{q}}$ in $\tilde{N}$ joining $\tilde{u}(\tilde{q})$ and $\tilde{v}(\tilde{q})$, such that $Z_i(\tilde{u}(\tilde{q})) = d\tilde{u}|_{\tilde{q}}(\tilde{E}_i)$, $Z(\tilde{v}(\tilde{q})) = d\tilde{v}|_{\tilde{q}}(\tilde{E}_i)$ and $\left\langle {}^{\tilde{N}}R(Z_i, \dot{\gamma}_{\tilde{q}})\dot{\gamma}_{\tilde{q}}, Z_i\right\rangle_{\tilde{N}} \equiv 0$ along $\tilde{\gamma}_{\tilde{q}}$. Moreover, $d(\mathrm{dist}_{\tilde{N}}(\tilde{j})) = 0$.
In particular, if ${}^N\mathrm{Sect} < 0$, Z_i is proportional to $\dot{\gamma}_{\tilde{q}}$ for each $i = 1, \dots, m$.

By the homotopy assumption, for each $q \in \partial M$ and any $\tilde{q} \in P_M^{-1}(q)$ we have $\tilde{u}(\tilde{q}) = \tilde{v}(\tilde{q})$, i.e., $\tilde{j}(\tilde{q}) \in \tilde{D}$. In particular, this implies that $\hat{r}^2(j)|_{\partial M} = 0$, and, since $d\hat{r}^2 = 2\hat{r}d\hat{r}$,

$$X|_{\partial M} = 0.$$

Then, applying the divergence theorem,

$$\int_M \operatorname{div} X dV_M = 0. \tag{12}$$

On the other hand, by the p-harmonicity of u and v and by the isometry property of the coverings projections,

$$\begin{aligned} {}^M \operatorname{div} X|_q &= \operatorname{tr}_M {}^{\hat{N}} \operatorname{Hess} \hat{r}^2|_{j(q)} (dj, J) + d\hat{r}^2|_{j(q)} (\operatorname{div} J|_q) \\ &= \operatorname{tr}_M {}^{\hat{N}} \operatorname{Hess} \hat{r}^2|_{j(q)} (dj, J) \\ &= \operatorname{tr}_{\tilde{M}} {}^{\tilde{N}\times\tilde{N}} \operatorname{Hess} \operatorname{dist}^2_{\tilde{N}} |_{\tilde{j}(\tilde{q})} \left(d\tilde{j}|_{\tilde{q}}, \tilde{J}|_{\tilde{q}} \right) \end{aligned} \tag{13}$$

for each $q \in M$ and any $\tilde{q} \in P_M^{-1}(q)$. By Lemma 3.1 we thus get $\operatorname{div} X \geq 0$ and (12) implies $\operatorname{div} X \equiv 0$. Thus (13) holds with the equality sign and the equality conditions in Lemma 3.1 give $d\left(\operatorname{dist}_{\tilde{N}}\right)(d\tilde{u}, d\tilde{v}) \equiv 0$. Since $\operatorname{dist}_{\tilde{N}}(\tilde{u}, \tilde{v})|_{\partial\tilde{M}} = 0$ we get $\tilde{u} \equiv \tilde{v}$ and, projecting on M, $u \equiv v$.
To conclude the proof, let us remark that in general relations (12) and (13) has to be considered in the weak sense. Lemma 7 in [V] proves the weak validity of (13), i.e.

$$-\int_M \left[d\hat{r}^2|_j \circ J\right] ({}^M\nabla\eta) = \int_M \eta \ \operatorname{tr}_M {}^{\hat{N}} \operatorname{Hess} \hat{r}^2|_{j(q)} (dj, J) \tag{14}$$

for all $\eta \in C_0^\infty(M)$. Moreover we can choose a 1-parameter family of smooth cut-off functions $\{\eta_\epsilon\}$ compactly supported in $\operatorname{int}(M)$ such that $\sup_M |\nabla\eta_\epsilon| = O(\epsilon^{-1})$ as $\epsilon \to 0$ and $\eta_\epsilon(q) = 1$ for all $q \in M$ satisfying $\operatorname{dist}_M(q, \partial M) > \epsilon$. Since $X|_{\partial M} \equiv 0$, X is continuous and

$$\operatorname{Vol}_M(\{q \in M : \operatorname{dist}_M(q, \partial M) \leq \epsilon\}) = O(\epsilon)$$

as $\epsilon \to 0$, applying (14) with $\eta = \eta_\epsilon$ and letting $\epsilon \to 0$, we can conclude that the LHS of (14) tends to 0. In some sense this gives a weak version of (12). On the other hand by Lemma 3.1, we can apply monotone convergence to the RHS of (14) to get

$$\int_M \operatorname{tr}_M {}^{\hat{N}} \operatorname{Hess} \hat{r}^2|_{j(q)} (dj, J) = 0.$$

□

References

[Au] T. Aubin, *Some nonlinear problems in Riemannian geometry.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998. xviii+395 pp.

[Bu] F. Burstall, Harmonic maps of finite energy from non-compact manifolds. *Jour. London Math. Soc.* **30** (1984), 361–370.

[DF] F. Duzaar and M. Fuchs, On removable singularities of p-harmonic maps. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **7** (1990), no. 5, 385–405.

[DF2] F. Duzaar and M. Fuchs, Existence and regularity of functions which minimize certain energies in homotopy classes of mappings. *Asymptotic Anal.* **5** (1991) 129–144.

[DGK] F. Duzaar, J.F. Grotowski and M. Kronz, Partial and full boundary regularity for minimizers of functionals with nonquadratic growth. (English summary) *J. Convex Anal.* **11** (2004), no. 2, 437–476.

[Ga] A. Gastel, Regularity theory for minimizing equivariant (p-)harmonic mappings. *Calc. Var. Partial Differential Equations* **6** (1998), 329–367.

[Fu] M. Fuchs, p-harmonic obstacle problems. I. Partial regularity theory. *Ann. Mat. Pura Appl.* **156** (1990), 127–158.

[Fu2] M. Fuchs, p-harmonic obstacle problems. III. Boundary regularity. *Ann. Mat. Pura Appl.* **156** (1990), 159–180.

[FR] A. Fardoun and R. Regbaoui, Heat flow for p-harmonic maps between compact Riemannian manifolds. *Indiana Univ. Math. J.* **51** (2002), 1305–1320.

[FR2] A. Fardoun and R. Regbaoui, Heat flow for p-harmonic maps with small initial data. *Calc. Var. Partial Differential Equations* **16** (2003), 1–16.

[FR3] A. Fardoun and R. Regbaoui, Regularity and uniqueness of p-harmonic maps with small range. *Geom. Dedicata* **164** (2013), 259–271.

[Ham] R. Hamilton, *Harmonic maps of manifolds with boundary.* Lecture notes, Mathematics, No. **471** Springer, Berlin, Heidelberg, New York (1975).

[HL] R. Hardt and F.-H. Lin, Mappings minimizing the L^p norm of the gradient. *Comm. Pure Appl. Math.* **40** (1987), no. 5, 555–588.

[Hat] A. Hatcher, *Algebraic topology.* Cambridge University Press, Cambridge, 2002. xii+544 pp.

[Hi] M.W. Hirsch, *Differential topology.* Graduate Texts in Mathematics, **33** Springer-Verlag, New York, 1994.

[HPV] I. Holopainen, S. Pigola and G. Veronelli Global comparison principles for the p-Laplace operator on Riemannian manifolds. *Potential Anal.* **34** no. 4 (2011), p. 371–384.

[Hu] N. Hungerbuhler, Global weak solutions of the p-harmonic flow into homogeneous space. *Indiana Univ. Math. J* **45** (1996), 275–288.

[Le] J. M. Lee, *Introduction to smooth manifolds.* Graduate Texts in Mathematics, **218**. Springer-Verlag, New York, 2003.

[Ma] A.-M. Matei, Gap phenomena for p-harmonic maps. *Annals of Global Analysis and Geometry* **18** (2000), 541–554.

[Mi] M. Misawa, Existence and regularity results for the gradient flow for p-harmonic maps. *Electron. J. Differential Equations* 1998, No. 36, 17 pp.

[Mu] J.R. Munkres, *Elementary differential topology.* Annals of Mathematics Studies, No. **54**. Princeton University Press, Princeton, N.J. 1966.

[N] N. Nakauchi, Regularity of minimizing p-harmonic maps into the sphere. *Nonlinear Analysis* **47** (2001), 1051–1057.

[PRS] S. Pigola, M. Rigoli and A.G. Setti, Constancy of p-harmonic maps of finite q-energy into non-positively curved manifolds. *Math. Z.* **258** (2008), no. 2, 347–362.

[PV1] S. Pigola and G. Veronelli, On the homotopy class of maps with finite p-energy into non-positively curved manifolds. *Geom. Dedicata* **143** (2009), 109–116.

[PV2] S. Pigola and G. Veronelli, *On the Dirichlet problem for p-harmonic maps II: Cartan-Hadamard targets with special structure.* preprint (2013).

[SU1] R. Schoen and K. Uhlenbeck, A regularity theory for harmonic maps. *J. Differential Geom.* **17** (1982), no. 2, 307–335.

[SU2] R. Schoen and K. Uhlenbeck, Boundary regularity and the Dirichlet problem for harmonic maps. *J. Differential Geom.* **18** (1983), no. 2, 253–268.

[SY1] R. Schoen and S.T. Yau, Compact group actions and the topology of manifolds with nonpositive curvature. *Topology* **18** (1979), 361–380.

[SY2] R. Schoen and S.T. Yau, *Lectures on Harmonic Maps.* Lectures Notes in Geometry and Topology, Volume II, International Press.

[T] K. Takegoshi, A maximum principle for P-harmonic maps with L^q finite energy. (English summary) *Proc. Amer. Math. Soc.* **126** (1998), no. 12, 3749–3753.

[V] G. Veronelli, A global comparison theorem for p-harmonic maps on Riemannian manifolds. *J. Math. Anal. Appl.* **391** (2012) 335–349.

[We1] S.W. Wei, The minima of the p-energy functional. *Elliptic and parabolic methods in geometry (Minneapolis 1994)*, pp. 171–203. A K Peters, Wellesley (1996).

[We2] S.W. Wei, Representing homotopy groups and spaces of maps by p-harmonic maps. *Indiana Univ. Math. J.* **47** (1998), no. 2, 625–670.

[WY] S.W. Wei and C.-M. Yau, Regularity of p-energy minimizing maps and p-superstrongly unstable indices. (English summary) *J. Geom. Anal.* **4** (1994), no. 2, 247–272.

[Wh] B. White, Homotopy classes in Sobolev spaces and the existence of energy minimizing maps. *Acta Math.* **160** (1988), no. 1-2, 1–17.

[Whd] J. H. C. Whitehead, On C^1-complexes. *Ann. of Math.* **41** (1940), 809–824.

[XY] Y.L. Xin and Y.H. Yang, Regularity of p-harmonic maps into certain manifolds with positive sectional curvature. *J. Reine Angew. Math.* **466** (1995), 1–17.

Dipartimento di Scienza e Alta Tecnologia, Università degli Studi dell'Insubria,, via Valleggio 11, I-22100 Como, ITALY

E-mail address: stefano.pigola@uninsubria.it

Université Paris 13, Sorbonne Paris Cité, LAGA, CNRS (UMR 7539), 99, avenue Jean-Baptiste Clément F-93430 Villetaneuse - FRANCE

E-mail address: giona.veronelli@gmail.com