Maximum and comparison principles at infinity on Riemannian manifolds

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0 Introduction

In the sixties H. Omori, [Om], studying minimal immersions into cones of the Euclidean space, introduced a global version of the maximum principle on any Riemannian manifold $(M, \langle , \rangle)$ whose sectional curvature is bounded from below and also provided examples where it fails to hold. When stated for the Laplace-Beltrami operator and with the obvious meaning of the symbols, the global maximum principle sounds as follows: for every bounded above $u \in \mathcal{C}^2(M)$ there is a sequence $\{x_n\} \subset M$ such that, for each n,

$$(1) \qquad i)\ u(x_n) > \sup_M u - \frac{1}{n}, \quad ii)\ |\nabla u|(x_n) < \frac{1}{n}, \quad iii)\ \Delta u(x_n) < \frac{1}{n}.$$

This new idea was taken up by S.T. Yau who, in a series of papers some in collaboration with S.Y.Cheng, see e.g. [Ya1], [ChYa], refined the principle for the Laplace-Beltrami operator and applied it to find elegant solutions to a series of geometric problems, most notably, the Schwarz lemma for holomorphic maps between Kahler manifolds. Besides the variety of results they obtained with this global principle, most of which by a PDE reformulation of the problem, a decisive contribution of Cheng and Yau lies in the way they were able to re-proved it. Unlike Omori approach, the far reaching version of their technique is very elementary and quite analytic; it makes an essential use of an old idea that in the thirties led Ahlfors to discover his deep version of the classical Pick-Schwarz lemma. Since then, the Ahlfors-Yau technique have become a versatile tool to solve a large amount of problems both of geometric and of analytic flavour: by way of example, the Calabi-Bernstein problem for closed, maximal, space-like hypersurfaces of the Lorentz space and the completeness of the affine metric (conjectured again by Calabi); in the same spirit, this approach recently led to formulate sharp a-priori estimates for solutions of partial differential inequalities which lie, e.g, on the base of the study of pointwise conformal deformations of Riemannian metric in the non-compact setting.

The papers of Omori, Yau, and Cheng-Yau also opened the way to a number of problems which arose in connection with the new maximum principle and which we may collect into three categories:

1. Find sharp forms of the maximum principle in relation with the geometry of the manifold;

2. Extend the maximum principle to differential operators other than the Laplacian;

3. Introduce some relaxed forms of the maximum principle.

Some of the contributions of this thesis go along the directions outlined in the second and in the third point (the first one being quite definitely dealt in a recent paper of A. Ratto, M. Rigoli and A.G. Setti, [RaRiSe]). As far as the second point is concerned, we note that Cheng-Yau proof makes an essential use of the linearity of the Laplace-Beltrami operator and thus their technique cannot be implemented when dealing with differential operators such as the mean curvature one. In these cases a new approach is needed. We are able to obtain a version of the Omori-Yau maximum principle at infinity for a large class of non-linear differential operators collected under the name of φ-Laplacians. Our approach is somewhat related to a paper of Redheffer, [Re]. Before stating the result, we introduce some terminology.

Let $(M, \langle,\rangle)$ denote a geodesically complete, non-compact, connected Riemannian manifold of dimension $\dim M = m$. Let $\varphi \in \mathcal{C}^0([0,+\infty)) \cap \mathcal{C}^1((0,+\infty))$ be a real valued function satisfying the structural conditions

$$\text{i) } \varphi(0)=0; \quad \text{ii) } \varphi'(t)>0,\ \forall t>0; \quad \text{iii) } \varphi(t)\le At^{\delta},\ \forall t\in[0,\varepsilon)$$

for some constants $A, \delta, \varepsilon > 0$ with possibly $\varepsilon = +\infty$. The φ-Laplacian of $u \in \mathcal{C}^1(M)$ is the divergence-form differential operator defined by

$$L_{\varphi}u = div\left(|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u\right).$$

Of course, if the vector field in brackets is not $\mathcal{C}^1$, then the divergence must be understood in distributional sense. In this respect, note that the vector field may fail to be $\mathcal{C}^1$ at points where $\nabla u = 0$ even if u is assumed to be $\mathcal{C}^2$. As important natural examples we mention the p-Laplacian, $div(|\nabla u|^{p-2}\nabla u)$, $p > 1$, corresponding to $\varphi(t) = t^{p-1}$ and the generalized mean curvature operator, $div\left(\nabla u/(1+|\nabla u|^2)^{\alpha}\right)$, $\alpha > 0$, corresponding to $\varphi(t) = t/(1+t^2)^{\alpha}$.

Under various assumptions on φ and in the Euclidean setting the φ-Laplacian has been studied by many authors over the past two decades. We mention in particular the seminal papers of J. Serrin and Serrin and L.A. Peletier [Se1], [PeSe], where Liouville type theorems are established. See also the recent paper of M. Rigoli and A.G. Setti [RiSe]. Here is our general maximum principle at infinity.

Theorem 1 ([PiRiSe2]) *Having fixed an origin $o \in M$, set $r(x) = dist_M(x,o)$ and assume that the radial Ricci curvature of M satisfies*

$$Ricci(\nabla r, \nabla r) \ge -(m-1)B^2G(r(x)) \quad \text{on } M$$

for some constant $B > 0$ and some positive, non-decreasing $G \in \mathcal{C}^1([0,+\infty))$ such that

$$\sqrt{G(t)} \le z(t)^{\delta}$$

where $\delta > 0$ is one of the structural constants of φ and $z(t) \in \mathcal{C}^1(+\infty)$ is positive, non-decreasing with

$$\frac{1}{z(t)} \notin L^1(+\infty).$$

Let $u \in \mathcal{C}^2(M)$ be such that $u^* = \sup_M u < +\infty$ and assume that the vector field $|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u$ is of class $\mathcal{C}^1$ on M. Then there exists a sequence $\{x_k\} \subset M$ with the following properties

$$\textit{i) } u(x_k) > u^* - \frac{1}{k}, \quad \textit{ii) } |\nabla u|(x_k) < \frac{1}{k}, \quad \textit{iii) } L_{\varphi}u(x_k) < \frac{1}{k}$$

for each $k \in \mathbb{N}$.

Coming back to the problems collected in the three categories above, as for the third point, we note that in a number of geometric situations a relaxed form of the Omori-Yau principle suffices. In particular, the gradient condition often plays no role. Thus, we are naturally led to introduce the following definition: the Riemannian manifold $(M, \langle , \rangle)$ is said to satisfy the weak maximum principle at infinity if for every $u \in \mathcal{C}^2(M)$, conditions (1) i) and iii) above hold along an appropriate sequence. It happens that this definition is very deep as we shall briefly explain in a moment. Recall that M is said to be Brownian complete if the (intrinsic) explosion time of the Brownian particle on M is almost surely infinite. A comprehensive account on the subject can be found in the nice survey paper of Grigor'yan, [Gr2]. We are able to prove the following result

Theorem 2 ([PiRiSe1]) $(M, \langle , \rangle)$ *is Brownian complete if and only if it satisfies the weak maximum principle at infinity.*

The discovery of this equivalence displays several advantages. On one hand, we have a new tool to investigate stochastic properties of M; in the other direction, we can obtain geometric and analytic results under very weak assumptions on the manifold. Nevertheless, it even suggests generalizations to new situations such as to the analysis of non-linear differential operators. Examples along all these lines are given.

We mentioned above that the Omori-Yau principle is successfully used to study global properties of solutions of PDEs on Riemannian manifolds. Especially to conclude uniqueness of solutions another basic tool is represented by comparison principles. When we deal with linear operators the task reduces to compare with a fixed solution, typically the null one. In this work we concentrate on the non-linear setting and establish comparison results over unbounded domains of a Riemannian manifold. Because of their global nature, the geometry of the underlying manifold turns out to play a decisive role.

Comparison principles over unbounded domains of $\mathbb{R}^2$ for the mean curvature operator have a long tradition starting from the pioneering works of Bers and Langevin and Rosenberg concerning the exterior Plateau problem. Recently they have been generalized in various ways to study prescribed mean curvature graphs in the Euclidean 3-space. All these results can be considered as L^∞-type comparisons in that they give uniqueness provided the distance between the graphs does not growth too fast with respect to the "volume growth" of the domain.

Our contribution to the theory is three-folds: by means of a new technique

1. We extend the L^∞-comparison principles, alluded above, to unbounded domains of complete Riemannian manifolds of any dimension;

2. We obtain L^p-versions that are new even in the Euclidean contest;

3. We deal with operators more general than the mean curvature one thus obtaining, in particular, global comparisons for the p-Laplacian, $p \geq 2$.

By way of example we state our L^p-result in the special situation of the mean curvature operator.

Theorem 3 ([PiRiSe]) *Let $(M, \langle , \rangle)$ be a complete, non-compact, connected Riemannian manifold of dimension $\dim M = m$. Having fixed an origin $o \in M$ denote by ∂B_r the geodesic sphere centered at o of radius $r > 0$. Let $\Omega \subset M$ be an unbounded domain with non-empty boundary $\partial\Omega$ and let $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ satisfy*

$$\begin{cases} div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = div\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) & on\ \Omega \\ u = v\ \ on\ \partial\Omega. \end{cases}$$

Assume that, for some $1 < q < +\infty$,

$$\frac{1}{\int_{\partial B_s \cap \Omega} |u - v|^q} \notin L^1(+\infty).$$

Then $u = v$ on Ω.

We conclude with a further remark on the mean curvature operator. The above mentioned comparison principles at infinity, from a different viewpoint, can be used to obtain asymptotic L^p-type estimates for minimal graphs defined on complete manifolds. For instance in Theorem 3 take Ω an exterior domain and $v \equiv 0$. If u is a non-null solution of the Dirichlet problem

$$\begin{cases} div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \text{ on } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

then

$$\frac{1}{\int_{\partial B_s} |u|^q} \in L^1(+\infty).$$

The L^∞-case goes similarly and gives a lower estimate of the type

$$\frac{\max_{\partial B_r} u}{\int^r \frac{ds}{vol\partial B_s}} \geq \delta > 0, \quad r >> 1$$

for some constant $\delta > 0$ provided

$$\frac{1}{vol\partial B_s} \notin L^1(+\infty).$$

One may ask about the possibility of an upper asymptotic estimate that would complete the above picture. We try to answer the question by considering exterior graphs over (essentially) Cartan-Hadamard manifolds. Precisely, we prove the following result

Theorem 4 *Let* $(M, \langle , \rangle)$ *be an* m*-dimensional manifold with a pole* $o \in M$*. Set* $r(x) = dist\,(x, o)$ *and denote by* $B_r, \partial B_r$ *the geodesic ball and sphere centered at* o *and of radius* $r > 0$*. Assume that the radial sectional curvature of* M *satisfies*

$$^{M}K_{rad} \leq 0.$$

Let $u \in \mathcal{C}^2\left(M \backslash \bar{B}_R\right) \cap \mathcal{C}^0\left(M \backslash B_R\right)$, $R > 0$, *be such that*

$$div\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) \leq 0.$$

Then

$$\limsup_{r \to +\infty} \frac{\min_{\partial B_r} u}{\log r} \leq R.$$

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1 The linear setting

This Chapter aims to present new developments in the framework of the maximum principle at infinity of Omori and Yau.

In Section 1, besides the historical notes, we prove a very general version of this global principle that gather in a single statement the previously known ones. Some examples of its use are also given.

In Section 2 we introduce a weak form of the maximum principle at infinity, we show its usefulness in solving some geometric and analytic problems and we give geometric conditions that guarantee its validity.

In Section 3 we briefly survey some concepts of completeness of a Riemannian manifold and we prove that the weak maximum principle is equivalent to the Brownian completeness of the underlying manifold. This, in particular, gives a new meaning to the previous section.

1.1 The Omori-Yau maximum principle

In 1938, L.V. Ahlfors [Ah] generalized the classical Pick-Schwarz lemma of complex analysis stating that holomorphic functions from the complex unit disc into itself contract the hyperbolic lengths. He understood that the core of Pick-Schwarz lemma lies in the comparison of conformal metrics with appropriate bounds on the respective Gaussian curvatures. This point of view enabled him to establish the contractivity property of holomorphic functions from the Poincarè disc into a general Riemann surface and to obtain a large amount of applications, e.g., to give a sharp estimate of the Bloch constant. The further extension to higher dimensions and the generalization to more general domains will take several years.
Here is the precise statement of Ahlfors result (in a simplified form):

Let the complex unit disc $D_1 = \{z \in \mathbb{C} : |z| < 1\}$ *be endowed with its Poincarè metric* $dp^2(z) = 4|dz|^2/(1-|z|^2)^2$ *of constat Gaussian curvature* -1. *If* $ds^2(z) = \lambda^2(z)|dz|^2$ *is a second conformal metric with Gaussian curvature* K_{ds^2} *satisfying* $K_{ds^2} \le -1$ *then* $\lambda(z) \le 2/(1-|z|^2)$

The proof is a simple but clever application of what we shall call through the paper "the classical maximum principle" (equivalently "the finite maximum principle") and which we can state as follows: let u be a $\mathcal{C}^2$, real valued function on a Riemannian manifold $(M, \langle,\rangle)$. If u attains its maximum at some point $x_0 \in M$ then

$$\text{(2)} \qquad \text{(i) } u(x_0) = \max_M u; \quad \text{(ii) } |\nabla u|(x_0) = 0; \quad \text{(iii) } \Delta u(x_0) \le 0,$$

where $|.|, \nabla, \Delta$ represent respectively the norm, the gradient and the Laplace-Beltrami operator computed with respect to $\langle,\rangle$. Clearly, another useful version of the classical maximum principle can be obtained by replacing condition (iii) with the stronger

$$\text{(iii)' Hess}u\,(x_0) \leq 0 \tag{3}$$

in the sense of quadratic forms. In this case we will speak of the finite maximum principle for the Hessian.

As a matter of facts, Ahlfors argument uses just condition (iii) and the linearity of the Laplace-Beltrami operator. In fact, his proof goes as follows.

Consider an exhaustion of D_1 by concentric discs D_r of radius $r \nearrow 1^-$ and define on each D_r the smooth functions $w_r(z) = 2r/(r^2 - |z|^2)$ and $\lambda_r(z) = \log(\lambda(z)/w_r(z))$. Note that $dp_r^2(z) = w_r^2(z)\,|dz|^2$ is nothing but the Poincarè metric of D_r so that $\Delta_{\mathbb{R}^2} \log w_r = w_r^2$; furthermore, because of the assumption on the curvature of the metric ds^2, it holds $\Delta_{\mathbb{R}^2} \log \lambda \geq \lambda^2$. As a consequence

$$\Delta_{\mathbb{R}^2} \lambda_r \geq \lambda^2 - w_r^2, \text{ on } D_r.$$

Since $\lambda_r|_{\partial D_r} = -\infty$, λ_r must attain its absolute maximum at some interior point $z_0 \in D_r$. By the classical maximum principle we thus deduce that $\Delta_{\mathbb{R}^2} \lambda_r(z_0) \leq 0$ which in turn, according to the above inequality, implies $\lambda_r(z_0) \leq 0$. This latter yields $\lambda_r(z) \leq 0$ on B_r as z_0 is the maximum of the function. Thus

$$\lambda(z) \leq w_r(z) \text{ on } D_r.$$

To conclude let the radius r tend to 1^-.

About forty years later, Ahlfors idea of using the finite maximum principle as explained above will become a great source of inspiration for S.T. Yau. Following the historical flow, in 1953 H. Omori, [Om], studying minimal immersions into cones of the Euclidean space introduced a global version of the maximum principle which has its root in the next simple observation: if the $\mathcal{C}^2$-function $u : \mathbb{R} \to \mathbb{R}$ is bounded above then, there exists a sequence $\{x_n\} \subset \mathbb{R}$ along which $u(x_n) > \sup_{\mathbb{R}} u - \frac{1}{n}$, $u'(x_n) < \frac{1}{n}$, $u''(x_n) < \frac{1}{n}$, for every $n \in \mathbb{N}$. Omori established a version of this principle on any Riemannian manifold $(M, \langle , \rangle)$ whose sectional curvature is bounded from below and also provided examples where his global form of the maximum principle fails to hold. When stated for the Laplace-Beltrami operator the global maximum principle sounds as follows: for every bounded above $u \in \mathcal{C}^2(M)$ there is a sequence $\{x_n\} \subset M$ such that, for each n,

$$\text{(i) } u(x_n) > \sup_M u - \frac{1}{n}, \quad \text{(ii) } |\nabla u|(x_n) < \frac{1}{n}, \quad \text{(iii) } \Delta u(x_n) < \frac{1}{n}. \tag{4}$$

The Hessian counterpart can be obtained replacing condition (iii) with

$$\text{(iii)' Hess}u\,(x_n) < \frac{1}{n} \langle , \rangle_{x_n} \tag{5}$$

in the sense of quadratic forms. In the sequel we shall refer to (4), (5) as the Omori-Yau maximum principle at infinity. Looking at its statement, this new principle can be really considered as a maximum principle at infinity in that it allows u to reach its supremum "at infinity" where, in a sense, the usual conditions stated in the classical maximum principle are guaranteed.

The proof provided by Omori was of very geometrical flavor. Yau in the seventies realized the great relevance of Omori result and in a series of papers, some in collaboration with S.Y. Cheng, applied it to find elegant solutions to a series of geometric problems, most notably an extension of the Ahlfors-Pick-Schwarz lemma to Kahler manifolds alluded above, [Ya3].

Cheng and Yau, see e.g. [Ya1], [ChYa], gave a simplified new proof of the principle and generalized it to new situations. Their argument is quite analytic and essentially based on the old Ahlfors use of the finite maximum principle. This is the reason why we can say that, in this route, one moves from finite to infinity.

The present section aims to illustrate Ahlfors-Yau technique by proving a very general version of the maximum principle at infinity, see Theorem 8 below. To better appreciate its importance we shall also give some applications to solve problems of both analytic and geometric nature, see Theorems 15 and 18.

First of all we focus our attention on conditions (4) (i) and (ii). It is a general fact that, given any $\mathcal{C}^2$, bounded above function u on a complete manifold $(M, \langle,\rangle)$, we can always find a sequence along which these conditions are met. Indeed

Proposition 5 *Let $(M, \langle,\rangle)$ be a Riemannian manifold and let $u \in \mathcal{C}^2(M)$ be such that $u^* < +\infty$. Given $\varepsilon > 0$, let $y \in M$ satisfy $u(y) > u^* - \varepsilon^2$ and suppose that the closed ball $\overline{B_\varepsilon(y)}$ be compact. Then, there exists $x \in \overline{B_\varepsilon(y)}$ with the properties*

$$(6) \qquad i)\ u(x) \geq u(y) \quad and \quad ii)\ |\nabla u(x)| \leq \varepsilon.$$

Proof. We let γ be the maximal integral curve of ∇u defined in $a < 0 < b$, such that $\gamma(0) = y$. Suppose first that $b < +\infty$. Then $\gamma([0, b])$ cannot lie in the compact set $\overline{B_\varepsilon(y)}$. A proof of this known fact is given in Lemma 6 below. Therefore, there exists some $t_0 \in [0, b)$ such that $\gamma(t_0) \in \partial B_\varepsilon(y)$.

On noting that

$$\frac{d}{dt} u(\gamma(t)) = \langle \nabla u(\gamma(t)), \dot{\gamma}(t)\rangle = |\nabla u(\gamma(t))|\, |\dot{\gamma}(t)|\,,$$

integrating both sides on $[0, t_0]$ we obtain

$$(7) \qquad u(\gamma(t_0)) - u(y) = \int_0^{t_0} |\nabla u(\gamma(t))|\, |\dot{\gamma}(t)|\, dt.$$

The validity of (6) can now be easily concluded. Indeed, if, for each $t \in [0, t_0]$ we had $|\nabla u(\gamma(t))| > \varepsilon$ then

$$u(\gamma(t_0)) - u(y) > \varepsilon \int_0^{t_0} |\dot{\gamma}(t)|\, dt = \varepsilon\ l\left(\gamma|_{[0,t_0]}\right) \geq \varepsilon^2$$

where $l\left(\gamma|_{[0,t_0]}\right)$ is the length of the segment $\gamma|_{[0,t_0]}$, and this would contradict $u(y) > u^* - \varepsilon^2$. Thus, there exists $\bar{t} \in [0, t_0]$ with $|\nabla u(\gamma(\bar{t}))| \leq \varepsilon$. Clearly, $x = \gamma(\bar{t})$ satisfies (6).

Now we consider the case $b = +\infty$. If γ goes out of $\overline{B_\varepsilon(y)}$ then we proceed as above, otherwise, that is $\gamma([0, +\infty)) \subset \overline{B_\varepsilon(y)}$, we reason as follows. Having fixed any $t_0 > 0$ we have the validity of (7). Thus, for each $t_0 > 0$, if $|\nabla u(\gamma(t))| > \varepsilon$ on $[0, t_0]$, then $u(y) \leq u(\gamma(t_0)) - t_0\varepsilon^2 \leq u^* - t_0\varepsilon^2$. Choosing $t_0 > 1$ we get a contradiction. Hence, there exists $\bar{t}$ for which $|\nabla u(\gamma(\bar{t}))| \leq \varepsilon$ and $\gamma(\bar{t}) \in \bar{B}_\varepsilon(y)$. We set $x = \gamma(\bar{t})$. ■

Lemma 6 *Let X be $\mathcal{C}^0$ vector field on M and γ its maximal integral curve defined on the interval $a < 0 < b$, such that $\gamma(0) = y$. If $b < +\infty$ and $\overline{B_\varepsilon(y)}$ is a compact set for some $\varepsilon > 0$, then $\gamma([0, b))$ cannot lie entirely in $\overline{B_\varepsilon(y)}$,*

Proof. Assume the contrary. Since

$$l(\gamma) = \int_a^b |\dot{\gamma}|(t)dt = \int_a^b |X|(\gamma(t))\,ds \leq \max_{\overline{B_\varepsilon(y)}} |X|(b-a) < +\infty$$

we deduce that $\gamma(t)$ must converge to some point $x_0 \in \overline{B_\varepsilon(y)}$. To see this, choose a sequence $\{t_{2n}\} \nearrow b^-$ as $n \to +\infty$. Compactness of $\overline{B_\varepsilon(y)}$ gives, up to a subsequence, $\gamma(t_{2n}) \to x_0$ as $n \to +\infty$ for some $x_0 \in \overline{B_\varepsilon(y)}$. If the limit $\lim_{t\to b^-} \gamma(t)$ did not exist we could find $\delta > 0$ sufficiently small and a sequence $\{s_{2n+1}\} \nearrow b^-$ such that $d(\gamma(s_{2n+1}), x_0) \geq \delta$. We could also suppose that $t_{2n} < s_{2n+1}$ for each n. Thus

$$\delta \leq d(x_0, \gamma(s_{2n+1})) \leq d(x_0, \gamma(t_{2n})) + d(\gamma(t_{2n}), \gamma(s_{2n+1}))$$

and for $n \geq N$ sufficiently large

$$d(\gamma(t_{2n}), \gamma(s_{2n+1})) \geq \frac{\delta}{2}.$$

As a consequence

$$l(\gamma) \geq \sum_{n\geq N} l(\gamma|_{[t_{2n}, s_{2n+1}]}) \geq \sum_{n\geq N} d(\gamma(t_{2n}), \gamma(s_{2n+1})) = +\infty$$

which is clearly impossible. We thus conclude that $\lim_{t\to b^-} \gamma(t) = x_0$, as claimed. Finally, since

$$\dot{\gamma}(t) = X(\gamma(t)) \to X(x_0), \text{ as } t \nearrow b^-$$

we obtain that $\gamma(t)$ can be extended past b, and this contradicts the maximality of (a, b). ■

Remark 7 For the conclusion of Proposition 5 to hold, some request on $B_\varepsilon(y)$ have to be considered. This is suggested by the following example. Let $M = \mathbb{R}^2 \setminus \{0\}$ with its canonical metric. We consider $u(x) = u(|x|) = e^{-|x|}$. Then $u \in C^\infty(M)$, $u^* = 1 = \lim_{|x|\to 0} u(x)$. On the other hand, $|\nabla u(x)| = e^{-|x|} \to 1$ as $|x| \to 0$.

Although on a every complete Riemannian manifold $(M, \langle,\rangle)$ we can always find a sequence $\{x_n\} \subset M$ satisfying (4) (i) and (ii) the next example shows that in general there might be no sequences satisfying all the three conditions at the same time. This points out the need of some geometrical assumption to prove the validity of the whole (4).

Let $(M, \langle,\rangle)$ be $\mathbb{R}^2$ with the metric, in polar coordinates,

$$\langle,\rangle = dr^2 + g(r)^2 d\theta^2 \tag{8}$$

with $d\theta^2$ the standard metric of S^1, $g \in \mathcal{C}^\infty([0,+\infty))$, $g(r) > 0$ for $r > 0$ and

$$g(r) = \begin{cases} r & \text{on } 0 \le r < 1 \\ r\,(\log r)^{1+\mu}\, e^{r^2 (\log r)^{1+\mu}} & \text{on } r > 3 \end{cases}$$

for some positive constant μ. We note that the behavior of g near 0 guarantees that (8) can be smoothly defined on all of $\mathbb{R}^2$. Obviously $\langle,\rangle$ is complete. We define

$$u(x) = u(r(x)) = \int_0^{r(x)} g(s)^{-1} \int_0^s g(t)dt\ ds.$$

Then, $u \in \mathcal{C}^2(M)$, $\Delta u \equiv 1$ on M, and, since $\mu > 0$, $u^* < +\infty$. In this case property (4) (iii) cannot hold. Note that in this example for the Gaussian curvature K and the volume growth of the geodesic ball B_r we respectively have

$$K(r) = -\frac{g''}{g}(r) \sim -c^2 r^2 (\log r)^{2(1+\mu)} \quad \text{as } r \to +\infty$$

for some constant $c > 0$, and

$$vol(B_r) \sim \frac{1}{2} e^{r^2 (\log r)^{1+\mu}} \quad \text{as } r \to +\infty.$$

The following result is a generalization of Cheng and Yau, [ChYa], [Ya1].

Theorem 8 *Let $(M, \langle,\rangle)$ be a Riemannian manifold and assume that there exists a non-negative $\mathcal{C}^2$ function γ satisfying the following requirements*

$$\gamma(x) \to +\infty \quad \textit{as } x \to \infty \tag{9}$$

$$\exists A > 0 \textit{ such that } |\nabla\gamma| \le A\gamma^{1/2} \textit{ off a compact set} \tag{10}$$

$$\exists B > 0 \textit{ such that } \Delta\gamma \le B\gamma^{1/2} G(\gamma^{1/2})^{1/2} \textit{ off a compact set} \tag{11}$$

where G is a smooth function on $[0,+\infty)$ satisfying

$$\begin{array}{ll} \textit{i) } G(0) > 0 & \textit{ii) } G'(t) \ge 0 \quad \textit{on } [0,+\infty) \\ \textit{iii) } G(t)^{-1/2} \notin L^1(+\infty) & \textit{iv) } \limsup_{t\to+\infty} \frac{tG(t^{1/2})}{G(t)} < +\infty. \end{array} \tag{12}$$

(Condition (9) means that for each $\eta > 0$ there is a compact set $K = K(\eta) \subset M$ such that $\gamma(x) > \eta$ whenever $x \notin K$.) Then, given any function $u \in \mathcal{C}^2(M)$ with $u^* = \sup_M u < +\infty$, there exists a sequence $\{x_n\}_n \subset M$ with the properties

$$\text{(13)} \qquad i)\ u(x_k) > u^* - \frac{1}{k}; \quad ii)\ |\nabla u(x_k)| < \frac{1}{k}; \quad iii)\ \Delta u(x_k) < \frac{1}{k}$$

for each $k \in \mathbb{N}$. If, instead of (11), we assume that

$$\text{(14)} \qquad \exists B > 0 \text{ such that } Hess(\gamma) \le B\gamma^{1/2} G(\gamma^{1/2})^{1/2} \langle , \rangle \text{ off a compact set}$$

in the sense of quadratic forms, we can strengthen conclusion (13) iii) to

$$Hess(u)(x_k) < \frac{1}{k} \langle , \rangle .$$

Proof. We define the function

$$\varphi(t) = e^{\int_0^t \sqrt{G(s)} ds}$$

and note that $\varphi(t)$ is well defined, smooth, positive and it satisfies

$$\text{(15)} \qquad \varphi(t) \to +\infty \quad \text{as } t \to +\infty.$$

We record, for future use, that

$$\varphi'(t) = G(t)^{-1/2} \varphi(t), \quad \varphi''(t) \le G(t)^{-1} \varphi(t)$$

and therefore,

$$\text{(16)} \qquad \left(\frac{\varphi'(t)}{\varphi(t)} \right)^2 - \frac{\varphi''(t)}{\varphi(t)} \ge 0$$

and, using assumption (12) iv),

$$\text{(17)} \qquad \frac{\varphi'(t)}{\varphi(t)} \le c \left(tG \left(t^{1/2} \right) \right)^{-1/2}$$

for some constant $c > 0$. Next, we fix a point $p \in M$ and, $\forall k \in \mathbb{N}$, we define

$$\text{(18)} \qquad f_k(x) = \frac{u(x) - u(p) + 1}{\varphi(\gamma(x))^{1/k}}.$$

Then $f_k(p) = 1/\varphi(\gamma(p))^{1/k} > 0$. Moreover, since $u^* < +\infty$ and $\varphi(\gamma(x)) \to +\infty$ as $x \to \infty$, we have $\limsup_{x \to \infty} f_k(x) \le 0$. Thus, f_k attains a positive absolute maximum at $x_k \in M$. Iterating this procedure, we produce a sequence $\{x_k\}$. We begin by showing that

$$\text{(19)} \qquad \limsup_{k \to +\infty} u(x_k) = u^*.$$

To prove the claim assume by contradiction that there exists $\hat{x} \in M$ such that

$$u(\hat{x}) > u(x_k) + \delta$$

for some $\delta > 0$ and for each $k \geq k_0$ sufficiently large. If $\gamma(x_k) \to +\infty$ as $k \to +\infty$, on a subsequence, for each k such that $\gamma(x_k) > \gamma(x)$ we have

$$f_k(\hat{x}) = \frac{u(\hat{x}) - u(p) + 1}{\varphi(\gamma(x))^{1/k}} > \frac{u(x_k) - u(p) + 1 + \delta}{\varphi(\gamma(x_k))^{1/k}} > f_k(x_k)$$

contradicting the definition of x_k. If $\{x_k\}$ lies in a compact set, then up to passing to a subsequence, $\{x_k\} \to \bar{x}$ so that

$$u(\hat{x}) \geq u(\bar{x}) + \delta.$$

On the other hand, from $f_k(x_k) \geq f_k(\hat{x})$ we deduce

$$u(\bar{x}) - u(p) + 1 = \lim_{k \to +\infty} f_k(x_k) \geq \lim_{k \to +\infty} f_k(\hat{x}) = u(\hat{x}) - u(p) + 1$$

showing that

$$u(\hat{x}) \leq u(\bar{x}),$$

a contradiction. This proves (19) and, by passing to a subsequence if necessary, we may assume that

$$\lim_{k \to +\infty} u(x_k) = u^*. \tag{20}$$

Again, if $\{x_k\}$ remains in a compact set, then $x_k \to \bar{x} \in M$ as $k \to +\infty$ and u attains its absolute maximum. At $\bar{x}$ we then have

$$u(\bar{x}) = u^*, \quad |\nabla u(\bar{x})| = 0, \quad Hess(u)(\bar{x}) \leq 0.$$

In this case the sequence $z_k = \bar{x}$, for each k, clearly satisfies all the requirements. We only need to consider the case when $x_k \to \infty$ so that, according to (9), $\gamma(x_k) \to +\infty$. Since f_k attains a positive maximum at x_k we have

$$\text{i) } (\nabla \log f_k)(x_k) = 0; \quad \text{ii) } Hess(\log f_k)(x_k) \leq 0. \tag{21}$$

A computation that uses (21) i), ii) yields

$$\nabla u(x_k) = \frac{1}{k}(u(x_k) - u(p) + 1)\frac{\varphi'(\gamma(x_k))}{\varphi(\gamma(x_k))}\nabla\gamma(x_k) \tag{22}$$

and

$$Hess(u)(x_k) \leq \frac{1}{k}(u(x_k) - u(p) + 1)\left\{\frac{\varphi'(\gamma(x_k))}{\varphi(\gamma(x_k))}Hess(\gamma)(x_k) + \right. \tag{23}$$
$$\left. + \left[\left(\frac{1}{k} - 1\right)\left(\frac{\varphi'(\gamma(x_k))}{\varphi(\gamma(x_k))}\right)^2 + \frac{\varphi''(\gamma(x_k))}{\varphi(\gamma(x_k))}\right] d\gamma \otimes d\gamma\right\}.$$

Whence, using (16) and the fact that $d\gamma \otimes d\gamma$ is positive semidefinite,

$$Hess\left(u\right)\left(x_k\right) \leq \frac{\left(u(x_k)-u(p)+1\right)}{k}\left\{\frac{\varphi'\left(\gamma\left(x_k\right)\right)}{\varphi\left(\gamma\left(x_k\right)\right)}Hess\left(\gamma\right)\left(x_k\right) + \frac{1}{k}\left(\frac{\varphi'\left(\gamma\left(x_k\right)\right)}{\varphi\left(\gamma\left(x_k\right)\right)}\right)^2 d\gamma\otimes d\gamma\right\}.$$

Taking traces in (23) we also have

$$\Delta u\left(x_k\right) = \frac{\left(u(x_k)-u(p)+1\right)}{k}\left\{\frac{\varphi'\left(\gamma\left(x_k\right)\right)}{\varphi\left(\gamma\left(x_k\right)\right)}\Delta\gamma\left(x_k\right) + \frac{1}{k}\left(\frac{\varphi'\left(\gamma\left(x_k\right)\right)}{\varphi\left(\gamma\left(x_k\right)\right)}\right)^2 \left|\nabla\gamma\left(x_k\right)\right|^2\right\}. \tag{24}$$

Assume now that (11) and (14) hold so that they hold at x_k for sufficiently large k. Using (16) in (22) we have

$$\begin{aligned}\left|\nabla u\left(x_k\right)\right| &= \frac{u(x_k)-u(p)+1}{k}\cdot\frac{\varphi'\left(\gamma\left(x_k\right)\right)}{\varphi\left(\gamma\left(x_k\right)\right)}\left|\nabla\gamma\left(x_k\right)\right| \\ &\leq \frac{cA}{k}\left(u(x_k)-u(p)+1\right)\cdot\frac{\gamma\left(x_k\right)^{1/2}}{\gamma\left(x_k\right)^{1/2} G\left(\gamma\left(x_k\right)^{1/2}\right)^{1/2}} \\ &\leq \frac{cA}{k}\cdot\frac{u(x_k)-u(p)+1}{G\left(\gamma\left(x_k\right)^{1/2}\right)^{1/2}}\end{aligned}$$

for some constant $c>0$, and the RHS tends to zero as $k\to+\infty$.

From (10) we have

$$d\gamma\otimes d\gamma \leq A^2\gamma\left\langle ,\right\rangle$$

whence substituting in (23) and using (14) and (16), (17) yield

$$Hess\left(u\right)\left(x_k\right) \leq \frac{c}{k}\cdot\frac{u^*-u(p)+1}{G\left(\gamma\left(x_k\right)^{1/2}\right)^{1/2}}\left\langle ,\right\rangle$$

and again the RHS tends to zero as $k\to+\infty$. It is also clear that if we assume that (11) holds and we argue as above, (24) yields

$$\Delta u\left(x_k\right) \leq \frac{c}{k}\cdot\frac{u^*-u(p)+1}{G\left(\gamma\left(x_k\right)^{1/2}\right)^{1/2}}.$$

■

Remark 9 Following Ahlfors idea, Yau argument uses the structural properties of the Laplacian and the Hessian operators in their full strength. Thus the technique cannot be used in non-linear contexts e.g. when dealing with the mean curvature operator.

Remark 10 It is evident that the conclusion of the theorem holds provided we can guarantee that

$$\frac{u^* - u(p) + 1}{G\left(\gamma\left(x_k\right)^{1/2}\right)^{1/2}}$$

is bounded above. For suitable choices of G and γ this may be obtained replacing the assumption that u be bounded above with an assumption on the growth of u at infinity.

Remark 11 The proof shows that we need γ to be C^2 only in a neighborhood of x_k. In the important situation where $\gamma(x) = r(x)^2$ is the square of the Riemannian distance from a fixed reference point o (see below) this is the case if x_k does not belong to the cut locus of o. Indeed one can deal with the cut-locus points using a trick of Calabi, [Ca], so that in fact, one may assume that $\gamma(x)$ is always C^2 in a neighborhood of x_k.

Remark 12 Especially significant examples of functions satisfying (12) are given by

$$G(t) = t^2 \prod_{j=1}^{N} \left(\log^{(j)}(t)\right)^2, \quad t \gg 1$$

where $\log^{(j)}$ stands for the j-th iterated logarithm.

At this point, it is important to underline that even if we have not use completeness in the proof of the Theorem, the two assumptions (9) and (10) imply it. Indeed, reasoning as in Cheng and Yau, [ChYa1], let $\sigma : [0, l) \to M$ be any divergent path, that is a path that eventually lies outside compact sets, parametrized by arc-length. From (10) we have

$$\frac{|\nabla\gamma|}{\gamma^{1/2}} \leq A \quad \text{outside } K$$

where $K \subset M$ is compact. We set $z(t) = \gamma(\sigma(t))$ on $[t_o, l)$ where t_0 has been chosen so that $\sigma(t) \notin K$ for each $t \in [t_0, l)$. Let $t \in [t_0, l)$, we have

$$\begin{aligned}\left|z(t)^{1/2} - z(t_0)^{1/2}\right| &= \left|\int_{t_0}^{t} \frac{z'(s)}{z(s)^{1/2}} ds\right| = \left|\int_{t_0}^{t} \frac{\langle\nabla\gamma\ (\sigma(s)), \dot{\sigma}(s)\rangle}{z(s)^{1/2}} ds\right| \\ &\leq \int_{t_0}^{t} \frac{|\nabla\gamma\ (\sigma(s))|}{\gamma(\sigma(s))^{1/2}} ds \leq A(t - t_0).\end{aligned}$$

By (9), $\gamma(x) \to +\infty$ as $x \to +\infty$. Thus, letting $t \to l^-$ from the above inequality we get $l = +\infty$. So divergent paths parametrized by arc-length have infinite length. This shows that the metric is complete. We shall come back to this later.

We describe two choices of γ and G suggested by geometry.

Example 13 *First choice.* Let $(M, \langle , \rangle)$ be a complete, non-compact, Riemannian manifold, let $o \in M$ be a reference point and denote by $r(x)$ the distance function from o. Set $\gamma(x) = r(x)^2$. Then, clearly, (9) and (10) are met, namely $\gamma(x) \to +\infty$ as $x \to \infty$ and $|\nabla\gamma| = 2\gamma^{1/2}$. Furthermore, γ is smooth within the cut-locus of o. Next, let G be a smooth function on $[0, +\infty)$ which we assume to be even at the origin, that is, $G^{(2k+1)}(0) = 0$ for each $k = 0, 1, 2, ...$ and satisfying the conditions listed in (12). If the radial sectional curvature of M, that is the sectional curvature of 2-planes containing ∇r, satisfies

$$ {}^{M}K_{rad} \geq -C^2\ G(r) \tag{25}$$

then, there exists $B > 0$ such that

$$ Hess(\gamma) \leq B\gamma^{1/2}G(\gamma^{1/2})^{1/2}\, \langle , \rangle \tag{26}$$

for every $x \in M$ within the cut-locus of o, with $r(x)$ sufficiently large. If we only assume that the radial Ricci curvature satisfies

$$ Ricci(\nabla r, \nabla r) \geq -C^2\ G(r) \tag{27}$$

then, (26) has to be replaced with

$$ \Delta\gamma \leq B\gamma^{1/2}G(\gamma^{1/2})^{1/2}. \tag{28}$$

Indeed, assuming for instance that (25) holds, by the Hessian comparison theorem (see [GrWu]) one has, within the cut locus of o,

$$ Hess(r) \leq \frac{\psi'}{\psi}\left(\langle , \rangle - dr \otimes dr\right)$$

where ψ is the solution of the initial value problem

$$\begin{cases} \psi'' = C^2\ G(r)\psi, & on\ [0, +\infty) \\ \psi(0) = 0 & \psi'(0) = 1. \end{cases}$$

(Note that the conditions on G imply that $\psi^{(2k)}(0) = 0$, for each $k = 0, 1, 2, ...$ which is necessary to construct a smooth model). Now, let h be the function defined by

$$ h(t) = D^{-1}\left\{ e^{D\int_0^t G(s)^{1/2}ds} - 1 \right\}.$$

Then, $h(0) = 0$, $h'(0) = 1$ and, for D sufficiently large

$$ h'' = \left(DG + \frac{1}{2}\frac{G'}{G^{1/2}}\right)\ e^{D\int_0^t G(s)^{1/2}ds} \geq C^2\ Gh$$

so that by Sturm comparison theorem

$$\frac{\psi'}{\psi} \leq \frac{h'}{h} = DG^{1/2} \frac{e^{D\int_0^t G(s)^{1/2}ds}}{e^{D\int_0^t G(s)^{1/2}ds} - 1} \leq DG^{1/2}.$$

Since

$$\begin{aligned} Hess(\gamma) &= 2rHess(r) + 2dr \otimes dr \\ &\leq 2rDG(r)^{1/2} \left(\langle , \rangle - dr \otimes dr\right) + 2dr \otimes dr \\ &= 2D\gamma^{1/2} G(r)^{1/2} (\gamma^{1/2}) \langle , \rangle + 2\left(1 - rDG(r)^{1/2}\right) dr \otimes dr \end{aligned}$$

the required conclusion follows easily. If we assume instead that (27) hold then (28) is achieved as above using the Laplacian comparison Theorem (see [GrWu]).

Example 14 *Second choice.* Let $f : (M^m, \langle , \rangle) \to (N^n, (,))$ be an isometric immersion. Let $p \in N$ and assume that either $cut(p) = \emptyset$ or, more generally, $f(M) \cap cut(p) = \emptyset$. Denote by ρ the distance function from p and let $\gamma = \rho^2 \circ f$. Then, γ is non-negative, C^2 and satisfies, since f is an isometric immersion,

(29) *i)* $|\nabla \gamma| \leq 2\gamma^{1/2}$; *ii)* $Hess(\gamma) = \left(Hess\left(\rho^2\right)\right) \circ (df \otimes df) + d\rho^2 \circ II$

where $II = \nabla df$ is the second fundamental tensor of the immersion. Taking traces in (29) ii) we obtain

$$^M\Delta\gamma = \sum_{i=1}^{m} Hess\left(\rho^2\right)\left(df(e_i) \otimes df(e_i)\right) + m\left(\nabla\rho^2, H\right)$$

where $m = \dim M$, $\{e_i\}_{i=1}^m$ is a local orthonormal basis and H is the mean curvature vector field of the immersion. Assume, for instance, that N is geodesically complete and that he radial sectional curvature of N with respect to p, satisfy

$$^N K_{rad} \geq -C^2 \ G(\rho)$$

where G is a smooth function with the properties listed in Theorem 8. By the Hessian comparison Theorem one has, corresponding to (26),

$$Hess(\rho^2) \leq B_1 \rho G(\rho)^{1/2} \langle , \rangle$$

outside a compact set and for some appropriate $B_1 > 0$. Thus, if we suppose that, off a compact set in M, it holds

$$|H| \leq B_2 G(\rho \circ f)^{1/2}$$

with $B_2 > 0$, we conclude that, for a suitable $B_3 > 0$,

$$^M\Delta\gamma \le B_3\ \ (\rho \circ f)\ \ G\,(\rho \circ f)^{1/2} = B_3\ \gamma^{1/2}\ G(\gamma^{1/2})^{1/2}.$$

In particular, assuming that the immersion is proper, we have $\rho \circ f(x) \to +\infty$ as $x \to \infty$ in M and all the conditions of Theorem 8 are met. Whence, the validity of conclusion (13). We have thus generalized a result of Kasue, [Ka].

We conclude the section by solving some problems with the aid of Theorem 8 to show its usefulness. We first consider minimal immersions into cones of the Euclidean space and obtain the following non-existence result which is originally due to Omori, [Om].

Theorem 15 *Let $\varphi : (M, \langle,\rangle) \to \mathbb{R}^n \setminus \{0\}$ be a connected, minimally immersed manifold. If $(M, \langle,\rangle)$ satisfies the Omori-Yau maximum principle for the Laplace-Beltrami operator, then $\varphi(M)$ is not contained in any non-degenerate cone of $\mathbb{R}^n$ with vertex at 0.*

By a non-degenerate cone of $\mathbb{R}^n$ we mean a cone which is strictly smaller than a half-space.

Proof. We reason by contradiction and we assume the existence of $b \in (0, 1)$, $v \in \mathbb{S}^{n-1}$ such that

$$\left\langle \frac{\varphi(x)}{|\varphi(x)|}, v \right\rangle \ge b, \quad \text{on } M. \tag{30}$$

Let

$$\tilde\varphi(x) = \varphi(x) - \langle \varphi(x), v\rangle\, v$$

so that

$$|\tilde\varphi(x)|^2 = |\varphi(x)|^2 - \langle \varphi(x), v\rangle^2 \tag{31}$$

From (30) and (31) we then have

$$\langle \varphi(x), v\rangle^2 - b^2\, |\tilde\varphi(x)|^2 \ge 0. \tag{32}$$

For any fixed $a \in (0, b/\sqrt{2}]$ we define

$$f_a(x) = -\langle \varphi(x), v\rangle + \sqrt{1 + a^2\, |\tilde\varphi(x)|^2}, \quad \text{on } M.$$

Then, independently of a, one has

$$f_a(x) \le 1, \quad \text{on } M. \tag{33}$$

Indeed, (33) is equivalent to

$$\sqrt{1 + a^2\, |\tilde\varphi(x)|^2} \le \langle \varphi(x), v\rangle + 1$$

that is,

$$\langle \varphi(x), v\rangle^2 - a^2 |\tilde{\varphi}(x)|^2 + 2\langle \varphi(x), v\rangle \geq 0. \tag{34}$$

Now, from (30) $\langle \varphi(x), v\rangle \geq 0$ and from (32)

$$\langle \varphi(x), v\rangle^2 - a^2 |\tilde{\varphi}(x)|^2 \geq \langle \varphi(x), v\rangle^2 - b^2 |\tilde{\varphi}(x)|^2 \geq 0,$$

hence the validity of (34).

Next, we fix a point $x_0 \in M$ and we define

$$\Omega_a = \{x \in M : f_a(x) \geq f_a(x_0)\} \neq \emptyset.$$

On Ω_a we then have

$$\begin{cases} -\langle \varphi(x), v\rangle + \sqrt{1 + a^2 |\tilde{\varphi}(x)|^2} \geq f_a(x_0) \\ \langle \varphi(x), v\rangle \geq b\,|\varphi(x)|. \end{cases}$$

Using these inequalities we get

$$\sqrt{1 + a^2 |\varphi(x)|^2} \geq \sqrt{1 + a^2 |\tilde{\varphi}(x)|^2} \geq b\,|\varphi(x)| + f_a(x_0). \tag{35}$$

We set

$$\Omega_a^{\pm} = \{x \in \Omega_a : b\,|\varphi(x)| + f_a(x_0) \gtrless 0\}$$

so that on Ω_a^- we have

$$|\varphi(x)| \leq \frac{-f_a(x_0)}{b} = \frac{\langle \varphi(x_0), v\rangle - \sqrt{1 + a^2 |\tilde{\varphi}(x_0)|^2}}{b} \leq \frac{\langle \varphi(x_0), v\rangle}{b}$$

while, squaring (35), we see that on Ω_a^+,

$$\begin{aligned} |\varphi(x)| &\leq \frac{-b\ f_a(x_0) + \sqrt{b^2 - a^2 + a^2 f_a(x_0)^2}}{b^2 - a^2} \\ &\leq \frac{2}{b}\left(|f_a(x_0)| + \sqrt{1 + |f_a\,(x_0)|^2}\right) \quad \text{on } \Omega_a^+ \end{aligned} \tag{36}$$

which is bounded on Ω_a^+ independently of $a \in (0, b/\sqrt{2}]$. Summarizing, $|\varphi|$ is bounded on Ω_a independently of $a \in (0, b/\sqrt{2}]$.

We define $f : M \to \mathbb{R}$ by setting

$$f(x) = f_a(x) - f_a(x_0).$$

Then f is non-negative precisely on the set Ω_a and $f(x_0) = 0$. Using what we have seen on $|\varphi|$ and recalling the definition of f_a we have that f is bounded on Ω_a independently of $a \in (0, b/\sqrt{2}]$. Whence, we conclude that f is bounded above on M independently of $a \in (0, b/\sqrt{2}]$.

We now fix a local Darboux frame $\{e_i\}$, $i = 1, ..., m = \dim M$ along φ and we denote with $\{\theta^i\}$ the dual coframe. Then, a simple computation that uses the minimality of φ yields

$$df = \left\{ \frac{a^2 \left[\langle e_i, \varphi \rangle - \langle \varphi, v \rangle \langle e_i, v \rangle \right]}{\left(a^2 \left| \tilde{\varphi} \right|^2 + 1 \right)^{1/2}} - \langle e_i, v \rangle \right\} \theta^i \tag{37}$$

$$\Delta f = \frac{ma^2}{\left(a^2 \left| \tilde{\varphi} \right|^2 + 1 \right)^{1/2}} - \frac{a^2 \sum_{i=1}^m \langle e_i, v \rangle^2}{\left(a^2 \left| \tilde{\varphi} \right|^2 + 1 \right)^{1/2}} - \frac{a^4 \sum_{i=1}^m \left(\langle e_i, \varphi \rangle - \langle \varphi, v \rangle \langle e_i, v \rangle \right)^2}{\left(a^2 \left| \tilde{\varphi} \right|^2 + 1 \right)^{3/2}}. \tag{38}$$

Using (37) into (38) we get

$$\Delta f = \frac{ma^2}{\left(a^2 \left| \tilde{\varphi} \right|^2 + 1 \right)^{1/2}} + \frac{\left(1 - a^2\right) \sum_{i=1}^m \langle e_i, v \rangle^2}{\left(a^2 \left| \tilde{\varphi} \right|^2 + 1 \right)^{1/2}} - \frac{1}{\left(a^2 \left| \tilde{\varphi} \right|^2 + 1 \right)^{1/2}} \left\{ \left| df \right|^2 + \frac{2a^2 \sum_{i=1}^m \langle e_i, v \rangle \left[\langle e_i, \varphi \rangle - \langle \varphi, v \rangle \langle e_i, v \rangle \right]}{\left(a^2 \left| \tilde{\varphi} \right|^2 + 1 \right)^{1/2}} \right\}$$

hence

$$\Delta f + \frac{\left| df \right|^2}{\left(a^2 \left| \tilde{\varphi} \right|^2 + 1 \right)^{1/2}} = \frac{ma^2}{\left(a^2 \left| \tilde{\varphi} \right|^2 + 1 \right)^{1/2}} + \frac{\left(1 - a^2\right) \sum_{i=1}^m \langle e_i, v \rangle^2}{\left(a^2 \left| \tilde{\varphi} \right|^2 + 1 \right)^{1/2}} - \frac{2a^2 \sum_{i=1}^m \langle e_i, v \rangle \langle e_i, \tilde{\varphi} \rangle}{\left(a^2 \left| \tilde{\varphi} \right|^2 + 1 \right)}. \tag{39}$$

Let A denote the RHS of (39). We shall show that, up to choosing $a \in (0, b/\sqrt{2}]$ sufficiently small,

$$A \geq \delta > 0 \quad \text{on } \Omega_a \tag{40}$$

for some δ. Indeed

$$
\begin{aligned}
A &\geq \frac{ma^2}{\left(a^2 |\tilde{\varphi}|^2 + 1\right)^{1/2}} + \frac{\left(1-a^2\right)\sum_{i=1}^m \langle e_i, v\rangle^2}{\left(a^2 |\tilde{\varphi}|^2 + 1\right)} - \frac{2a^2 \sum_{i=1}^m \langle e_i, v\rangle \langle e_i, \tilde{\varphi}\rangle}{\left(a^2 |\tilde{\varphi}|^2 + 1\right)} \\
&= \frac{ma^2}{\left(a^2 |\tilde{\varphi}|^2 + 1\right)^{1/2}} - \frac{a^2 \sum_{i=1}^m \langle e_i, v\rangle^2}{\left(a^2 |\tilde{\varphi}|^2 + 1\right)} + \frac{\sum_{i=1}^m \left(\langle e_i, v\rangle^2 - 2a^2 \langle e_i, v\rangle \langle e_i, \tilde{\varphi}\rangle\right)}{\left(a^2 |\tilde{\varphi}|^2 + 1\right)} \\
&= \frac{ma^2}{\left(a^2 |\tilde{\varphi}|^2 + 1\right)^{1/2}} - \frac{a^2 \sum_{i=1}^m \langle e_i, v\rangle^2}{\left(a^2 |\tilde{\varphi}|^2 + 1\right)} + \\
&+ \frac{\sum_{i=1}^m \left(\langle e_i, v\rangle - a^2 \langle e_i, \tilde{\varphi}\rangle\right)^2}{\left(a^2 |\tilde{\varphi}|^2 + 1\right)} - \frac{a^4 \sum_{i=1}^m \langle e_i, \tilde{\varphi}\rangle^2}{\left(a^2 |\tilde{\varphi}|^2 + 1\right)} \\
&\geq \frac{ma^2}{\left(a^2 |\tilde{\varphi}|^2 + 1\right)^{1/2}} - \frac{a^2 |v|^2}{\left(a^2 |\tilde{\varphi}|^2 + 1\right)} - \frac{a^4 |\tilde{\varphi}|^2}{\left(a^2 |\tilde{\varphi}|^2 + 1\right)} \\
&\geq \frac{ma^2}{\left(a^2 |\tilde{\varphi}|^2 + 1\right)^{1/2}} - a^2.
\end{aligned}
$$

We now use the fact that $|\varphi|$ is bounded on Ω_a independently of a to choose $a \in (0, b/\sqrt{2}]$ so small that, for some $0 < \varepsilon < m^2 - 1$,

$$
a^2 |\varphi|^2 < m^2 - 1 - \varepsilon, \quad \text{on } \Omega_a.
$$

Thus

$$
A \geq a^2 \left(\frac{m}{(m^2 - \varepsilon)^{1/2}} - 1 \right) = \delta > 0
$$

and (40) holds with this choice of a and δ.

Now, (39) becomes

$$
\Delta f + \frac{|df|^2}{\left(a^2 |\tilde{\varphi}|^2 + 1\right)^{1/2}} \geq \delta. \tag{41}
$$

By the Omori-Yau maximum principle there exists a sequence $\{x_k\} \subset M$ such that

$$
|df(x_k)| < \frac{1}{k}, \quad \Delta f(x_k) < \frac{1}{k}
$$

for each $k \in \mathbb{N}$. Moreover, since $f(x_k) \to f^* = \sup_M f \geq 0$, without loss of generality, we can assume that $\{x_k\} \subset \Omega_a$. Hence, using (41) and the above we get

$$
\frac{1}{k} + \frac{1}{k^2} \frac{1}{\left(a^2 |\tilde{\varphi}(x_k)|^2 + 1\right)^{1/2}} \geq \delta > 0.
$$

Letting $k \to +\infty$ we finally obtain a contradiction. ■

Applying the above result and Example 14 we immediately obtain the following

Corollary 16 *A complete Riemannian manifold* $(M, \langle , \rangle)$ *does not admit a proper, isometric, minimal immersion into any non-degenerate cone of some Euclidean space* $\mathbb{R}^n$.

Remark 17 It is apparent from the proof, see (36), that the cone has to be strictly smaller than the half-space.

It is an open problem to give a version of the Theorem for a half-space. The famous Half-Space Theorem by Hoffman and Meeks, [HoMe], for 2-dimensional surfaces in $\mathbb{R}^3$ gives very subtle indications.

We now present a way to use the maximum principle to obtain analytic results by proving an "a-priori" estimate for a class of semilinear PDEs.

The following theorem appears in [RaRiSe] and extends results of Cheng-Yau [ChYa] and Motomyia [Mt].

Theorem 18 *Assume on* $(M, \langle , \rangle)$ *the validity of the maximum principle for the Laplace-Beltrami operator. Let* $u \in \mathcal{C}^2(M)$ *be a solution of the differential inequality*

$$\Delta u \geq \varphi\left(u, |\nabla u|\right) \tag{42}$$

with $\varphi(t, y)$ *continuous in* t, $\mathcal{C}^2$ *in* y *and such that*

$$\frac{\partial^2 \varphi}{\partial t^2}(t, y) \geq 0. \tag{43}$$

Set $f(t) = \varphi(t, 0)$. *Then a sufficient condition to guarantee that*

$$u^* = \sup_M u < +\infty$$

is the existence of a continuous function F *positive on* $[a, +\infty)$ *for some* $a \in \mathbb{R}$, *satisfying the following inequalities*

$$\left\{ \int_a^t F(s) ds \right\}^{-1/2} \in L^1(+\infty) \tag{44}$$

$$\limsup_{t \to +\infty} \frac{\int_a^t F(s) ds}{t F(t)} < +\infty \tag{45}$$

$$\liminf_{t \to +\infty} \frac{f(t)}{F(t)} > 0 \tag{46}$$

$$\text{(47)} \qquad \liminf_{t\to+\infty} \frac{\left\{\int_a^t F(s)ds\right\}^{1/2}}{F(t)} \frac{\partial\varphi}{\partial y}(t,0) > -\infty.$$

Furthermore, in this case, we have

$$\text{(48)} \qquad f(u^*) \leq 0.$$

Proof. Let $g \in \mathcal{C}^2(\mathbb{R})$ be a function to be specify later, satisfying

$$\text{(49)} \qquad g \geq 1, \quad g' > 0, \quad g'' \neq 0 \quad \text{on } \mathbb{R}.$$

Since $1/g(u)$ is bounded below on M, the validity of the maximum principle (in the obvious form of a minimum principle) yields the existence of a sequence $\{x_k\} \subset M$ such that

$$\text{(50)} \qquad \lim_{k\to+\infty} \frac{1}{g(u(x_k))} = \inf_M \frac{1}{g(u)}$$

$$\text{(51)} \qquad \frac{1}{k} > \left|\nabla\left(\frac{1}{g(u)}\right)(x_k)\right| = \frac{|g'(u)|\,|\nabla u|}{g(u)^2}(x_k)$$

$$\text{(52)} \qquad -\frac{1}{k} < \Delta\left(\frac{1}{g(u)}\right)(x_k) = -\frac{g''(u)\,|\nabla u|^2}{g(u)^2}(x_k) + \\ -\frac{g'(u)\Delta u}{g(u)^2}(x_k) + \frac{2g'(u)^2\,|\nabla u|^2}{g(u)^3}(x_k)$$

for each $k \in \mathbb{N}$. Multiplying (52) times

$$\frac{g'(u)^2}{g(u)^2\,|g''(u)|}(x_k)$$

and using $g \geq 1$, (42), (51) at x_k we obtain

$$\text{(53)} \qquad \frac{g'(u)^3\varphi(u,|\nabla u|)}{g(u)^4\,|g''(u)|} \leq \frac{g'(u)^2}{g(u)\,|g''(u)|}\frac{1}{k}\left(1+\frac{2}{k}\right) + \frac{1}{k^2}.$$

Next, we use the Taylor formula with respect to y and (43) to have

$$\varphi(u,|\nabla u|) \geq f(u) + \frac{\partial\varphi}{\partial y}(u,0)\,|\nabla u|\,.$$

Hence, the LHS in (53) is bounded below by

$$\frac{g'(u)^3 f(u)}{g(u)^4\,|g''(u)|} + \frac{g'(u)^3}{g(u)^4\,|g''(u)|}\frac{\partial\varphi}{\partial y}(u,0)\,|\nabla u|\,.$$

On the other hand, using (51) at x_k,

$$\frac{g'(u)^3}{g(u)^4\,|g''(u)|}\frac{\partial\varphi}{\partial y}(u,0)\,|\nabla u| \geq \frac{1}{k}W(u) \tag{54}$$

where we have set

$$W(u) = k\ \min\left\{0, \frac{1}{k}\frac{g'(u)^2}{g(u)^2\,|g''(u)|}\frac{\partial\varphi}{\partial y}(u,0)\right\}.$$

Summarizing, at x_k, we have

$$\frac{g'(u)^3}{g(u)^4\,|g''(u)|}f(u) + \frac{1}{k}W(u) \leq \frac{g'(u)^2}{g(u)\,|g''(u)|}\frac{1}{k}\left(1+\frac{2}{k}\right) + \frac{1}{k^2}. \tag{55}$$

We now choose g. Let $b \geq a$ and define $g(t)$ on $\mathbb{R}$ by requiring

$$g(t) = \int_b^t \frac{ds}{\left(\int_a^s F(r)dr\right)^{1/2}} + 2, \quad \text{on } [b,+\infty) \tag{56}$$

$g\,(t)$ increasing on $(-\infty, b]$ from 1 to 2 and $g''(t) \neq 0$ on $\mathbb{R}$. Then,

$$\limsup_{t\to+\infty} \frac{g'(t)^2}{g(t)\,|g''(t)|} < +\infty \tag{57}$$

$$\liminf_{t\to+\infty} \frac{g'(t)^3}{g(t)^4\,|g''(t)|} > 0. \tag{58}$$

Indeed, (57), (58) are respectively equivalent to

$$\limsup_{t\to+\infty} \frac{\left(\int_a^t F(r)dr\right)^{1/2}}{g(t)F(t)} < +\infty \tag{59}$$

$$\liminf_{t\to+\infty} \frac{f(t)}{g(t)^4 F(t)} > 0 \tag{60}$$

and since

$$g(t) \geq \frac{t-b}{\left(\int_a^s F(r)dr\right)^{1/2}}, \quad \text{for } t \gg 1,$$

condition (59) follows from (45) while (60) is a consequence of (44) and (46). Furthermore, by the definition of W, g in (56) and assumption (47) imply that

$$\liminf_{t\to+\infty} W(t) \geq -B^2 \tag{61}$$

for some constant $B > 0$.

To finish the proof, we now reason by contradiction and we assume that $u^* = +\infty$. Since g is increasing (50) gives

$$\lim_{k \to +\infty} u(x_k) = +\infty. \tag{62}$$

Letting $k \to +\infty$ in (55) and using (62), (58), (61) we obtain the desired contradiction. Hence $u^* < +\infty$. As for the conclusion $f(u^*) \leq 0$ we take the limit as $k \to +\infty$ in (55) and we note that, again, since g is increasing,

$$\lim_{k \to +\infty} u(x_k) = u^*.$$

■

1.2 The weak maximum principle

In the previous Section we gave some applications of the global maximum principle of Omori and Yau to solve problems of both geometric and analytic nature. It happens however that, in a number of situations, a relaxed form of the Omori-Yau principle suffices. In particular, the gradient condition may play no role. We illustrate these situations via some concrete examples. To this end, it is convenient and quite natural to introduce the following

Definition 19 *A Riemannian manifold* $(M, \langle , \rangle)$ *is said to satisfy the weak maximum principle at infinity for the Laplacian (weak maximum principle, for short) if for every* $u \in \mathcal{C}^2(M)$ *such that* $u^* = \sup_M u < +\infty$ *we find a sequence* $\{x_k\} \subset M$ *along which*

$$\text{(63)} \qquad i)\ u(x_k) > u^* - \frac{1}{k} \quad and \quad ii)\ \Delta u(x_k) < \frac{1}{k}.$$

When condition (63) ii) is replaced by

$$ii)'\ \mathrm{Hess}u(x_k) < \frac{1}{k} \langle , \rangle_{x_k}$$

in the sense of quadratic forms, we say that the weak maximum principle for the Hessian holds.

To begin with, we note that the weak maximum principle at infinity is tightly related to a-pripri estimates for bounded solutions of some elliptic PDEs. In fact, we have the following

Theorem 20 *Suppose that the Riemannian manifold* $(M, \langle , \rangle)$ *satisfies the weak maximum principle for the Laplacian. For any* $f \in \mathcal{C}^0(\mathbb{R})$ *let* $u \in \mathcal{C}^2(M)$ *be a bounded above function satisfying, for some* $\gamma < \sup_M u < +\infty$,

$$\text{(64)} \qquad \Delta u \geq f(u) \ on\ \Omega_\gamma$$

where we have set $\Omega_\gamma = \{x \in M : u(x) > \gamma\}$. *Then* $f(\sup_M u) \leq 0$.

Proof. By contradiction, suppose that there exists f such that the differential inequality in (64) has a bounded solution u on Ω_γ, for some γ, satisfying

$$\text{(65)} \qquad f(\sup_M u) = c > 0.$$

By the weak maximum principle at infinity, we find a sequence $\{x_n\}$ along wich $u(x_n) > \sup_M u - \frac{1}{n}$ and $\Delta u(x_n) < \frac{1}{n}$. Up to taking n sufficiently large, we may assume that $\{x_n\} \subset \Omega_\gamma$ and therefore

$$\frac{1}{n} > \Delta u(x_n) \geq f(u(x_n)).$$

Whence, letting n tend to $+\infty$ and using (65) we reach the desired contradiction. ∎

Remark 21 We shall see in the next section that this property is in fact equivalent to the validity of the weak maximum principle.

We now derive some geometric consequences of Theorem 20. The next result relates the mean curvature of bounded isometric immersions with the smallest extrinsic radius. We recall that a geodesic ball $B_R(q)$ of a Riemannian manifold $(N, (\,,\,))$ is said to be regular if $cut(q) \cap B_R(q) = \emptyset$ and $\sup_{B_R(q)} {}^N K \leq \left(\frac{\pi}{2R}\right)^2$, where ${}^N K$ denotes the sectional curvatures of N.

Theorem 22 *Let $B_R(q) \subset (N, (\,,\,))$ be a regular geodesic ball with $\sup_{B_R(q)} {}^N K \leq K$ for some $K \in \mathbb{R}$, and let $\varphi : (M, \langle\,,\rangle) \to B_R(q)$ be a smooth map with tension field $\tau(\varphi)$ satisfying $|\tau(\varphi)| \leq \tau_o$ for some $\tau_o > 0$. Assume that $(M, \langle\,,\rangle)$ satisfies the weak form of the maximum principle at infinity for the Laplacian. Then, denoting by $e(\varphi)$ the energy density of φ, we have*

$$R \geq \begin{cases} K^{-1/2} \arctan\left(2K^{1/2}\tau_o^{-1} \inf_M e(\varphi)\right) & \text{if } K > 0 \\ 2\tau_o^{-1} \inf_M e(\varphi) & \text{if } K = 0 \\ (-K)^{-1/2} arctanh\left(2(-K)^{1/2}\tau_o^{-1} \inf_M e(\varphi)\right) & \text{if } K < 0. \end{cases}$$

Proof. We consider only the case $K < 0$, the other cases being similar. To simplify notation we assume that $K = -1$. Denote by ρ the distance function in $(N, (\,,\,))$ from the point q, and set $u = 1/2 \cosh(\rho \circ \varphi)$. A straightforward computation that uses the Hessian comparison theorem (see [GrWu]) yields

$$\Delta u \geq u\left[2e(\varphi) + \text{tahn}(\rho \circ \varphi)(\nabla\rho, \tau(\varphi))\right].$$

Since $u \geq 1/2$, and $-\tau_o \text{tanh} R \leq \tanh(\rho \circ \varphi)(\nabla\rho, \tau(\varphi))$ on M, we deduce that

$$\Delta u \geq \inf_M e(\varphi) - 1/2\, \tau_o \tanh R.$$

Now the required conclusion (with $K = -1$) follows from Theorem 20. ■

Remark 23 For the conclusion of Theorem 22 to hold it suffices that there exists a sequence $\{x_k\}$ in M such that

$$\Delta u(x_k) < 1/k \qquad \textit{for every } k.$$

We also note that if $\varphi : (M, \langle\,,\rangle) \to (N, (\,,\,))$ is an isometric immersion and $m = dim M$, then

$$2e(\varphi) = m \quad \textit{and} \quad \tau(\varphi) = mH,$$

where H denotes the mean curvature vector. This case has been considered in [Ka], assuming conditions that imply the weak maximum principle on the complete manifold $(M, \langle\,,\rangle)$.

Remark 24 Theorem 22 can also be applied to give a negative answer to a question of Calabi, (see [EeKo]), on the existence of complete minimal surfaces in $\mathbb{R}^3$ with bounded image. We point out that in a recent paper, N.S Nadirashvili

[Na] (see also the corrections in [CoRo]) has exhibited an example solving in the affirmative this long standing problem. However, the search of mild additional geometric assumptions under which the image $\varphi(M)$ is necessarily unbounded remains a challenging task. In this respect we quote the beautiful striking half-space theorem of Hoffman and Meeks [HoMe].

The next result is due to S. Goldberg, [Go]. We recall that a Riemannian manifold $(M^m, \langle , \rangle)$ is said to be (locally) conformally flat if, for each $x_0 \in M$ there is a coordinate chart $(U, x = (x_1, ..., x_m))$ centered at x_0 where we have the local expression $\langle , \rangle_x = \lambda(x)^2 \, can_{\mathbb{R}^m}$ for some $0 < \lambda \in \mathcal{C}^\infty(U)$. It is known that every two-dimensional manifold is locally conformally flat and that, in general, this fails to hold in higher dimensions.

Theorem 25 *Let $(M, \langle , \rangle)$ be a connected, locally conformally flat, Riemannian manifold of dimension $m \geq 3$. Assume the validity of the weak maximum principle at infinity. If the scalar curvature S of $(M, \langle , \rangle)$ is a positive constant and*

$$\|Ricci\|^2 \leq \frac{S^2}{m-1} - \varepsilon \tag{66}$$

for some small $\varepsilon > 0$ then $(M, \langle , \rangle)$ has constant sectional curvature.

Proof. Let F denote the square of the length of the traceless Ricci tensor

$$F = \left\| Ricci - \frac{S}{m} \langle , \rangle \right\|^2 .$$

Clearly, F is a smooth, non-negative, bounded above function. Moreover since M is conformally flat and S is constant, F obeys the following differential inequality (see [Go] and [Go1])

$$\frac{m-1}{2} \Delta F \geq F \left(S - \sqrt{m(m-1)F} \right).$$

Note that

$$\inf_M \left(S - \sqrt{m(m-1)F} \right) > 0$$

because of (66). Therefore from Theorem 20 we deduce $F \equiv 0$ which means that M is Einstein. This fact together with the conformal flatness give that M is of constant sectional curvature. ■

The following can be considered as a Schwarz-type lemma.

Theorem 26 *Let $(M, \langle , \rangle)$ be a Riemannian manifold with scalar curvature $s(x) \leq -\epsilon$ on M, for some $\epsilon > 0$. Assume that the weak maximum principle for the Laplacian holds on M. Let $\varphi : M \to M$ be a conformal immersion preserving the scalar curvature, and assume that, having set $\varphi^* \langle , \rangle = v \langle , \rangle$, we have $\inf_M v > 0$. Then φ is weakly distance increasing.*

Proof. We consider the case where $m \geq 3$, the case $m = 2$ being similar. Setting $v = u^{4/(m-2)}$, it is well known that the function u is a solution of the Yamabe equation

$$\Delta u = -\frac{m-2}{4(m-1)} s(x) u (u^{4/(m-2)} - 1) \quad \text{on } M, \tag{67}$$

where we have used the assumption that φ preserves the scalar curvature. We want to show that $\inf_M u \geq 1$. Assume the contrary, fix $1 > \delta > \inf_M u > 0$ and define

$$\Omega_{-\delta} = \{x \in M : u(x) < \delta\}.$$

Since $s(x) \leq -\varepsilon$ we have

$$\Delta u \leq -\frac{m-2}{4(m-1)} \varepsilon u \left(1 - \delta^{4/(m-2)}\right) \text{ on } \Omega_{-\delta}$$

and therefore, applying Theorem 20 gives $\inf_M u = 0$ contradicting the assumptions. ■

Remark 27 Theorem 26 can be completed by saying that, in the same assumptions on $(M, \langle , \rangle)$, a conformal quasi-isometry preserving the scalar curvature is necessarily an isometry. Indeed, once we know that $u \geq 1$, from (67) we deduce $\Delta u \geq \varepsilon \frac{(m-2)}{4(m-1)} u \left(u^{4/(m-2)} - 1\right)$ on M. Since we are assuming that $\sup_M u < +\infty$, application of Theorem 20 yields $u \leq 1$.

We now consider the weak form of the maximum principle at infinity for the Hessian. The next theorem is due to Jorge and Koutrofiotis, [JoKo]. It extends on non-compact manifolds classical non-existence results for low-codimensional isometric immersions into Euclidean spaces due to Kuiper and Chern and Tompkins.

Theorem 28 *Let $(M, \langle , \rangle)$ be an m-dimensional Riemannian manifold satisfying the weak maximum principle for the Hessian. Suppose that $(N, (,))$ is a manifold of dimension n with*

$$m + 1 \leq n \leq 2m - 1$$

and let ${}^N B_R(q)$ be a regular geodesic ball of radius R in N. Let $\varphi : M \to N$ be an isometric immersion with the property that $\varphi(M)$ lies inside ${}^N B_R(q)$ and let $A > 0$ be a constant.
a) If $\sup_{{}^N B_R(q)} {}^N K(\sigma) = A^2$ and $R < \frac{\pi}{2A}$ then

$$\inf_{{}^N B_R(q)} {}^N K(\sigma) + A^2 \cot^2(AR) \leq \sup_M {}^M K(\pi). \tag{68}$$

b) If $\sup_{{}^N B_R(q)} {}^N K(\sigma) = 0$ then

$$\inf_{{}^N B_R(q)} {}^N K(\sigma) + \frac{1}{R^2} \leq \sup_M {}^M K(\pi). \tag{69}$$

c) If $\sup_{{}^N B_R(q)}\ {}^N K(\sigma) = -A^2$ *then*

$$\inf_{{}^N B_R(q)}\ {}^N K(\sigma) + A^2 \coth^2(AR) \le \sup_M\ {}^M K(\pi). \tag{70}$$

To prove the Theorem, we shall make use of the following algebraic result due to Otsuki, [Ot].

Lemma 29 *Let* $\alpha : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^q$ *with* $q \le m-1$ *be a symmetric bilinear form such that* $\alpha(V,V) \ne 0$ *for each* $V \ne 0$*. Then there exist linearly independent vectors* X *and* Y *such that*

$$\alpha(X,X) = \alpha(Y,Y) \quad \textit{and } \alpha(X,Y) = 0 \tag{71}$$

Proof (of Theorem 28). We observe that if $\sup_M\ {}^M K(\pi) = +\infty$ then the estimates of the theorem are trivially satisfied. Thus, we assume $\sup_M$ ${}^M K(\pi) < +\infty$. We set $\rho(.) = dist_{(N,(,))}(.,q)$. Since φ is non-constant, there exists $p \in M$ such that $\varphi(p) \ne q$. Set $2\alpha = \rho(\varphi(p)) > 0$. Then, on the non-empty open set

$$\Omega_\alpha = \{x \in M : \rho(\varphi(x)) > \alpha\}$$

the function $\rho \circ \varphi$ is smooth because ${}^N B_R(q)$ is a regular ball.

According to cases a), b) and c) we let, respectively,

$$z(t) = \sin(At), \quad z(t) = t, \quad z(t) = \sinh(At).$$

We define

$$Z(t) = \int_0^t z(s)ds$$

and we consider

$$u = Z \circ \rho \circ \varphi.$$

Then, $u \in C^\infty(\Omega_\alpha) \cap C^0(M)$, u is bounded and its supremum u^* on M coincides with its supremum on Ω_α. We may therefore think of u as smoothly extended on all of M. According to the weak maximum principle for the Hessian, there exists a sequence $p_k \in \Omega_\alpha$ such that

$$\text{i) } u(p_k) > u(p) > 0; \quad \text{ii) Hess}u(p_k)(X,X) \le \frac{1}{k}\,|X|^2_{\langle,\rangle}, \ \forall X \in T_{p_k}M. \tag{72}$$

To simplify the writing we set $\rho_k = \rho(\varphi(p_k))$ and we observe that $\alpha < \rho_k \le R$. By the Hessian comparison theorem on the ball ${}^N B_R(q)$

$$\text{Hess}(\rho) \ge \frac{z'(\rho)}{z(\rho)}\{(,) - d\rho \otimes d\rho\}$$

so that

$$\text{Hess}(Z \circ \rho) \ge z'(\rho)\ \ (,) \qquad \text{on } {}^N B_R(q).$$

whence, using (72) ii), the fact that φ is an isometric immersion and Gauss Lemma we obtain

$$\begin{aligned}
\frac{1}{k}\,|X|^2_{\langle,\rangle} &\geq \mathrm{Hess}(u)(p_k)(X,X) = \mathrm{Hess}(Z\circ\rho\circ\varphi)(p_k)(X,X)\\
&= \mathrm{Hess}(Z\circ\rho)(p_k)(d\varphi\otimes d\varphi)(X,X) + (z\circ\rho\circ\varphi)\,(p_k)(\nabla\rho,\nabla d\varphi)(X,X)\\
&\geq z'(\rho_k)\,|X|^2_{\langle,\rangle} - z(\rho_k)\,|\nabla d\varphi(X,X)|_{(,)}
\end{aligned}$$

Dividing by $z(\rho_k)\,|X|^2_{\langle,\rangle}$ with $X \neq 0$ (note that $\rho_k > 0$ forces $z(\rho_k) > 0$) we finally obtain

$$\frac{|\nabla d\varphi(X,X)|_{(,)}}{|X|^2_{\langle,\rangle}} \geq \frac{z'(p_k)}{z(p_k)} - \frac{1}{k\ z(\rho_k)}. \tag{73}$$

Since $\rho_k > \alpha > 0$ we can suppose to have chosen k large enough so that the RHS of (73) is positive. Having fixed in this way p_k, we observe that $(\nabla d\varphi)\,(p_k) = II_{p_k}$ is the second fundamental form at p_k of the immersion φ; further

$$II_{p_k} : T_{p_k}M \times T_{p_k}M \to T_{p_k}M^{\perp}$$

where $\dim T_{p_k}M^{\perp} = n-m$ with $1 \leq n-m \leq m-1$ because of the assumptions on n and m. Since, according to (73), $II_{p_k}(X,X) \neq 0$, for each $0 \neq X \in T_{p_k}M$, we can apply Lemma 29 to deduce the existence of two linearly independent vectors $X, Y \in T_{p_k}M$ such that

$$II_{p_k}(X,X) = II_{p_k}(Y,Y) \quad \text{and} \quad II_{p_k}(X,Y) = 0. \tag{74}$$

Using (73) and (74) we have

$$\begin{aligned}
\left[\frac{z'(p_k)}{z(p_k)} - \frac{1}{k\ z(\rho_k)}\right]^2 &\leq \frac{|II_{p_k}(X,X)|_{(,)}\,|II_{p_k}(Y,Y)|_{(,)}}{|X|^2_{\langle,\rangle}\,|Y|^2_{\langle,\rangle}}\\
&\leq \frac{(II_{p_k}(X,X), II_{p_k}(Y,Y)) - |II_{p_k}(X,Y)|^2_{(,)}}{|X|^2_{\langle,\rangle}\,|Y|^2_{\langle,\rangle} - \langle X,Y\rangle^2}.
\end{aligned}$$

Let π be the 2-plane spanned by X and Y. Then, according to Gauss equations and the definition of sectional curvature we have

$${}^{M}K_{p_k}(\pi) - {}^{N}K_{\varphi(p_k)}\,(\varphi_*\pi) = \frac{(II_{p_k}(X,X), II_{p_k}(Y,Y)) - |II_{p_k}(X,Y)|^2_{(,)}}{|X|^2_{\langle,\rangle}\,|Y|^2_{\langle,\rangle} - \langle X,Y\rangle^2} \tag{75}$$

and thus

$${}^{N}K_{\varphi(p_k)}\,(\varphi_*\pi) + \left[\frac{z'(p_k)}{z(p_k)} - \frac{1}{k\ z(\rho_k)}\right]^2 \leq {}^{M}K_{p_k}(\pi). \tag{76}$$

Whence

$$\inf_{{}^{N}B_R(q)} {}^{N}K\,(\sigma) + \left[\frac{z'(p_k)}{z(p_k)} - \frac{1}{k\ z(\rho_k)}\right]^2 \leq \sup_{M} {}^{M}K(\pi). \tag{77}$$

Since the sequence $\{\rho_k\}$ is bounded we can suppose, without loss of generality, that $\rho_k \to \bar{R}$ with $0 < \alpha \leq \bar{R} \leq R$. Letting $k \to +\infty$ into (77) we obtain

$$\inf_{{}^N B_R(q)} {}^N K(\sigma) + \left[\frac{z'(\bar{R})}{z(\bar{R})}\right]^2 \leq \sup_M {}^M K(\pi).$$

But in all the three cases a),b),c), the quantity z'/z is decreasing and therefore

$$\inf_{{}^N B_R(q)} {}^N K(\sigma) + \left[\frac{z'(R)}{z(R)}\right]^2 \leq \sup_M {}^M K(\pi).$$

The result now follows immediately. ■

Remark 30 It should be noted that results on compact, isometric submanifolds of the Euclidean space can often be extended to the non-compact realm and in a more general ambient requiring small images and the validity of some form of the maximum principle at infinity.

Remark 31 We mention that exactly the same technique with a different choice of the geometric function Z yields similar curvature estimates for non-compact submanifolds into right cylinders of $\mathbb{R}^n$, see [HaKo].

Until now the validity of the weak maximum principle has been considered as an hypothesis. Our next task is to give sufficient conditions on the underlying manifold that insure its validity. Needless to say, all that we said in the previous section concerning the global maximum principle in its full strength applies in the present situation. Thus, in the geodesically complete setting, the existence of a proper immersion into some Euclidean space with controlled mean curvature, as well as lower bounds on the (Ricci, Sectional) curvatures suffice.

The following is a sufficient functional theoretic property.

Theorem 32 *Let $(M, \langle , \rangle)$ be a Riemannian manifold, and assume that M supports a C^2 function γ which tends to infinity at infinity, and satisfies the differential inequality*

$$\Delta\gamma \leq \lambda\gamma \qquad \textit{off a compact set} \tag{78}$$

for some $\lambda > 0$. Then the weak maximum principle for the Laplacian holds on $(M, \langle , \rangle)$.

Proof. Let u be a bounded above, C^2 function on M, and assume by contradiction that (63) does not hold. It follows that there exists $\bar{\epsilon}$ such that

$$\Delta u > \bar{\epsilon}, \tag{79}$$

on the set $\{x : u(x) > u^* - \bar{\epsilon}\}$. Note that this implies that u does not attain u^*.

Next we consider the function $u - c\gamma$, where $c > 0$ will be specified later. Note that by adding a constant to γ we may assume that γ be everywhere positive and that (78) hold on M. Since γ tends to $+\infty$ as x goes to infinity, $u - c\gamma$ attains a maximum m_c on a set Γ_c.

We claim that if c is sufficiently small, then, for every ξ_c in Γ_c we have

$$u(\xi_c) > u^* - \bar{\epsilon}/2 \qquad \text{and} \qquad c\lambda\gamma(\xi_c) < \bar{\epsilon}/2. \tag{80}$$

Indeed, let $\bar{x} \in M$ be such that $u(\bar{x}) > u^* - \eta/2$, and choose $c > 0$ small enough that $c\gamma(\bar{x}) \leq \eta/2$. Then

$$u^* \geq u^* - c\gamma(\xi_c) \geq (u - c\gamma)(\xi_c) \geq (u - c\gamma)(\bar{x}) \geq u^* - \eta,$$

whence $u(\xi_c) \geq u^* - \eta$ and $c\gamma(\xi_c) \leq \eta$, and the claim follows by choosing $\eta \leq \min(\bar{\epsilon}/2, \bar{\epsilon}/(2\lambda))$.

Thus, $\Gamma_c \subseteq \{x : u(x) > u^* - \bar{\epsilon}\}$ and (79) holds at every point of Γ_c. But, recalling that $u - c\gamma$ attains a maximum on Γ_c, and using (78) we have that

$$0 \geq \Delta(u - c\gamma)(\xi_c) \geq \Delta u(\xi_c) - c\lambda\gamma(\xi_c)$$

at every $\xi_c \in \Gamma_c$, whence, using (80),

$$\Delta u(\xi_c) \leq \bar{\epsilon}/2,$$

contradicting the assumption (79). ■

A minor modification of the above proof, yields the following version of Theorem 32 for the Hessian.

Theorem 33 *Assume that Riemannian manifold $(M, \langle , \rangle)$ supports a C^2 function γ which tends to infinity at infinity, and satisfies the differential inequality*

$$\text{Hess}\gamma \leq \lambda\gamma \langle , \rangle \qquad \textit{off a compact set} \tag{81}$$

in the sense of quadratic forms, for some $\lambda > 0$. Then the weak maximum principle for the Hessian holds on $(M, \langle , \rangle)$.

In connection with Theorem 32, it may be interesting to observe that the existence of a function γ satisfying the conditions listed in the statement is in fact equivalent with the apparently weaker requirement that there exist a function σ defined outside a compact set in M and satisfying

$$\sigma(x) \to +\infty \quad \text{as } x \to \infty, \qquad \Delta\sigma \leq f(\sigma), \tag{82}$$

where f is C^1 and positive for $t \gg 1$, $1/f \notin L^1(+\infty)$ and $\inf f'(t) > -\infty$ (note that any function f such that $f(t) \leq t \prod_{j=1}^{N} \log^{(j)} t$, for $t \gg 1$ where $\log^{(j)}$ is the jth iterated logarithm, satisfies the requirement).

Indeed, assume that such a function σ exists. By adding a constant to f we may suppose that f be defined and positive for $t \geq t_o > 0$ and there satisfies $f'(t) \geq -A + 1$ for some $A > 0$. Let ϕ be the function defined by the formula

$$\phi(t) = \exp\left(\int_{t_o}^{t} \frac{1}{f(s) + As} ds\right) \qquad t > t_o,$$

and set $\gamma = \phi(\sigma)$. Since $\sigma \to +\infty$ as $x \to \infty$, γ is defined outside a compact set in M, and since $1/f$ is not integrable at infinity, γ tends to infinity at infinity. Moreover, a simple computation shows that

$$\Delta\gamma \leq \phi(\sigma) - \frac{\phi(\sigma)}{(f(\sigma)+A\sigma)^2}\,(f'(\sigma)+A-1)\,|\nabla\sigma|^2.$$

Since the second term on the right hand side is positive by the assumption made on f', we conclude that $\Delta\gamma \leq \gamma$ as required.

We remark that the condition on the derivative of f may be relaxed considerably. For instance, the proof carries out in the same way if it is only assumed that $f'(t) \geq (-A+1)\log t$ for t large, provided ϕ is defined by

$$\phi(t) = \exp\left(\int_{t_o}^{t} \frac{1}{f(s)+As\log s}\right) ds \qquad t > t_o.$$

More generally, it is enough to require that $f'(t) \geq (-A+1)\prod_{j=1}^{N}\log^{(j)} t$ for some $N \geq 1$ and $A \geq 0$.

We also observe that the existence of a function γ satisfying the above requirements does not force the manifold to be geodesically complete. This should be compared with the observation following the proof of Theorem 8. Contrary to what happened there, in the present situation, no conditions are imposed on the gradient of the function γ, and this allows one to find functions satisfying the requirements even on non complete manifolds. For instance, if $M = \mathbb{R}^m \setminus \{0\}$, $m \geq 3$, with the usual Euclidean metric, then M is not geodesically complete, but the function $\gamma = |x|^2 + |x|^{2-m}$ satisfies the conditions.

We can apply Theorem 32 in the case where $(M, \langle,\rangle)$ is a model in the sense of Greene and Wu. If o denotes the origin of the model, the metric may be written in geodesic polar coordinates

$$\langle,\rangle = dr^2 + g(r)^2 d\theta^2 \qquad \text{on } M \setminus \{o\} \tag{83}$$

with $g \in C^\infty([0,+\infty))$, $g^{(2k)}(0) = 0$ $k = 0,1,\ldots$, $g'(0) = 1$, $g > 0$ on $(0,+\infty)$. Then

$$vol\partial B_r = g^{m-1}(r), \qquad volB_r = \int_0^r g(t)^{m-1}dt.$$

We define

$$\gamma(r) = \int_0^r \frac{volB_t}{vol\partial B_t}dt.$$

Since $\Delta r = (m-1)g'/g(r)$ in the complement of o, a straightforward computation gives

$$\Delta\gamma \equiv 1 \qquad \text{on } M. \tag{84}$$

It follows from Theorem 32 that if

$$\frac{volB_r}{vol\partial B_r} \notin L^1(+\infty), \tag{85}$$

then $(M, \langle , \rangle)$ is stochastically complete. On the other hand, if (85) does not hold, then γ is a bounded C^2 solution of (84) on M. Thus the weak maximum principle does not hold. We have therefore proved the following

Proposition 34 *A necessary and sufficient condition for the validity of the weak maximum principle for the Laplacian on a model manifold is that the volume growth condition (85) is satisfied.*

We conclude the section by noting that the weak maximum principle is preserved under compact deformations. This is the content of the next

Proposition 35 *Let $(M, \langle , \rangle)$ and $(N, (,))$ be noncompact Riemannian manifolds, and assume that there exist compact sets $A \subseteq M$ and $B \subseteq N$ and a Riemannian isometry $\varphi : M \setminus A \to N \setminus B$ which preserves divergent sequences in the ambient spaces, that is, $\{x_k\}$ diverges in M if and only if $\{\varphi(x_k)\}$ diverges in N. Then $(M, \langle , \rangle)$ satisfies the weak maximum principle for the Laplacian if and only if so does $(N, (,))$.*

Proof. Suppose that $(M, \langle , \rangle)$ is stochastically complete, and let u be a C^2 bounded functions on N. We are going to show that, for some sequence $\{y_k\} \subseteq N$, we have $u(y_k) \to u^* = \sup u$ and $\Delta_N u(y_k) \leq 1/k$. Without loss of generality, we may assume that u^* is not attained, and strictly positive. We let K_1, K_2 be two relatively compact domains in M such that $A \subseteq K_1 \subseteq \bar{K}_1 \subseteq K_2$. Choose a smooth cutoff function $\psi : M \to [0, 1]$ satisfying $\psi \equiv 0$ on K_1, $\psi \equiv 1$ on $M \setminus K_2$, and define a function $v \in C^2(M)$ by the formula

$$v = \begin{cases} \psi \; u \circ \varphi & \text{on } M \setminus A \\ 0 & \text{on } A. \end{cases}$$

We claim that $v^* = \sup v = u^*$ and that v^* is not attained. Indeed, by construction $u^* \geq v^*$. On the other hand, let $\{\bar{y}_k\}$ be a sequence in N such that $u(\bar{y}_k) \nearrow u^*$. Since u does not attain u^*, the sequence $\{\bar{y}_k\}$ is divergent. In particular, for k sufficiently large $\bar{y}_k$ lies outside B. By the assumption on φ, $\bar{x}_k = \varphi^{-1}(\bar{y}_k)$ is divergent in M, and therefore it lies eventually outside $\bar{K}_2$. Thus

$$v(\bar{x}_k) = \psi(\bar{x}_k)(u \circ \varphi)(\bar{x}_k) = u(\bar{y}_k),$$

showing that $v(\bar{x}_k) \nearrow u^*$ and $u^* = v^*$. Further, since $v(x) = 0$ on A, while $v(x) \leq u(\varphi(x)) < u^* = v^*$ on $M \setminus A$, v does not attain v^*, as claimed.

Now it is an easy matter to conclude the proof of the proposition. Since M is stochastically complete, there exists a sequence $\{x_k\} \subseteq M$ such that, for every k, $v(x_k) \nearrow v^*$, and $\Delta v(x_k) < 1/k$. Since v^* is not attained, the sequence diverges in M, and we may assume that $x_k \in M \setminus \bar{K}_2$ for every k. Having set $y_k = \varphi(x_k)$, it follows from the definition of v and the fact that φ is a Riemannian isometry, that

$$\begin{aligned} &u(y_k) = v(x_k) \nearrow v^* = u^* \\ &\Delta_N u(y_k) = \Delta_M v(x_k) < 1/k. \end{aligned}$$

Thus, the weak maximum principle holds on $(N, (\,,\,))$. Repeating the same argument with M and N interchanged (note that $\varphi^{-1} : N \setminus B \to M \setminus A$, is a Riemannian isometry which maps divergent sequences to divergent sequences) shows that if N satisfies the weak maximum principle at infinity so does M. ■

The following examples illustrate the role of the assumptions of Proposition 35.

Example 36 Let $(M, \langle\,,\,\rangle) = (\mathbb{R}^2 \setminus \{0\}, can)$, $(N, (\,,\,)) = (\mathbb{R}^2, can)$, where "*can*" denotes the canonical flat metric; $A = \emptyset$, $B = \{0\}$. Let also $\varphi : M \setminus A \to N \setminus B$ be the identity map. Then φ is clearly an isometry, but it does not preserve diverging sequences. Indeed, the sequence $\{(1/n, 1/n)\}$ diverges in M, while $\varphi((1/n, 1/n))$ converges in N.

Example 37 Let $(M, \langle\,,\,\rangle) = (\mathbb{R}^2, can)$, $(N, (\,,\,)) = (\mathbb{R}^2 \setminus B_1(0)), can)$, $A = B_1(0) \cup \{(2, 0)\}$, $B = \{(2, 0)\}$, and let again $\varphi : M \setminus A \to N \setminus B$ be the identity map. Then φ is a Riemannian isometry, $(M, \langle\,,\,\rangle)$ satisfies the weak maximum principle, while $(N, (\,,\,))$ does not. To verify the last statement, consider the function $u(x) = |x|^{-2} \in C^{\infty}(N)$. With a slight abuse of notation we still denote by u its extension to $\mathbb{R}^2 \setminus B_{1/2}(0)$. Then $u^* = \sup_N u = 1 = u|_{\partial B_1(0)}$. On the other hand $\Delta u \geq c > 0$ on $\partial B_1(0)$ showing that the weak maximum principle cannot hold.
Again, φ does not preserve diverging sequences.

Example 38 If $(M, \langle\,,\,\rangle)$ and $(N, (\,,\,))$ are geodesically complete, then any isometry $\varphi : M \setminus A \to N \setminus B$ automatically preserves diverging sequences.

Example 39 If φ preserves relative compactness in the ambient spaces, that is, $U \subseteq M \setminus A$ is relatively compact in M if and only if $\varphi(U)$ is relatively compact in N, then φ preserves diverging sequences.

1.3 Stochastic completeness and the weak maximum principle

Before coming into the main result of the section, see Theorem 40 below, we take a pause of reflection and, looking at what we shown in the previous pages, we outline a synthetic view of maximum principles at infinity. The exposition will be of informal character and give us the opportunity to rise some natural questions that meet the interest of our personal taste.

Riemannian manifolds can be studied from different viewpoints, e.g., that of geometry, that of classical and stochastic analysis etc., and clearly, in all these cases, one can deal with theories both of local and of global nature. A quite natural place where developing global (Riemannian!) theories is a compact manifold. In the non-compact setting, however, the search of globality leads us to require some further property on the underlying manifold: for instance, that the ambient is not a piece of a bigger one, or that our objects live for infinite time or something else that compactness automatically guarantees. We collect properties of these types under the (improper) name of completeness of the Riemannian manifold. Thus, we naturally come into different concepts of completeness each oriented to the viewpoint we married. The possible interplays between them is a fascinating subject.

In the geometric direction, a first step towards a global theory is to require that the manifold, say $(M, \langle , \rangle)$, is non-extensible, i.e., it is not isometric to a proper open subset of a second connected ambient $(N, (,))$. However, it happens that the class of non-extensible manifolds is too large. A basic object in Riemannian geometry is represented by geodesic paths. Although there always exists a (unique) small geodesic segment issuing from a point with a given speed, such geodesic could explode in finite time. Think for instance to the Riemannian covering M of the standard punctured plane $\mathbb{R}^2 \setminus \{0\}$. It is a non-extensible manifold and the line segment $\gamma(t) = (1-t, 0)$, $-\infty \leq t < 1$, lifts to a geodesic $\tilde{\gamma}(t)$ of M that explodes to infinity near $t = 1^-$.
When each geodesic path can be extended on all of the real line, the manifold is said to be geodesically complete. We know that geodesic completeness implies non-extensibility and that, according to the above mentioned example, the converse does not hold.
The classical theorem by Hopf and Rinow states that this very Riemannian completeness is equivalent to the metric one once we endow $(M, \langle , \rangle)$ with the natural distance induced by the Riemannian metric. Moreover, this latter turns out to be equivalent to the Heine-Borel property (closed and bounded subsets are compact). An immediate but nevertheless deep consequence of these characterizations is the robustness of the concept of geodesic completeness, i.e., its invariance under quasi-isometries.
Every smooth manifold possesses a complete Riemannian metric. This can be seen in different ways e.g. using the Whitney imbedding theorem, or defining first a smooth metric via a partition of unity argument and next deforming it to

a complete point-wise conformal metric (thus showing, in particular, that every Riemannian metric has a complete representative in its conformal class).

Similarly to what happens in geometry, also a global study from the analytic viewpoint suggests that the manifold should be non-extensible but this seems to be not at all satisfactory. Looking at compactness one realizes that a compact Riemannian manifold is characterized by the validity of the classical maximum principle, i.e., for each $u \in \mathcal{C}^2(M)$ there is a point $x_0 \in M$ where $u(x_0) = \sup_M u$, $\nabla u(x_0) = 0$ and $\Delta u(x_0) \leq 0$ (although very interesting, here and in the generalization below, we do not consider its Hessian version). Indeed if M was non-compact then we could find an infinite sequence $A = \{x_n\}$ without limit points. Thus, defining $u : A \to \mathbb{R}$ by $u(x_n) = n$, we could extend u to a smooth unbounded function on all of M. The proof can now be easily completed.
In the non-compact setting one can consider a similar situation and require that, possibly "at infinity", a form of the above maximum principle holds true. Since we cannot insure that smooth functions are bounded, we have the following natural formulation: for each bounded above $u \in \mathcal{C}^2(M)$ there is a sequence $\{x_n\} \subset M$ along which $u(x_n) > \sup_M u - \frac{1}{n}$, $|\nabla u(x_n)| < \frac{1}{n}$ and $\Delta u(x_n) < \frac{1}{n}$.
In the literature, this is known as the Omori-Yau maximum principle at infinity for the Laplacian. As above it represents a global property of M which displays deep applications in the study of the qualitative behavior of solutions of PDEs as well as in geometry, see e.g. Theorem 18 above.
In a sense, the Omori-Yau principle is a kind of analytic completeness: in particular, it forces the manifold to be non-extensible. To see this, suppose M is an open subset of a connected second manifold N. Fix $o \in \partial M$ and a small geodesic ball $B_\varepsilon(o)$. Next, define $r(x) = dist_N(x, o)$ and consider the function $u(x) = e^{-r(x)}$ on $B_\varepsilon(o) \setminus \{o\}$. Extend u on all of $N \setminus \{o\}$ by means of a cut-off function. Thus $u \leq 1$ on M equality holding at o. Since for each sequence $\{x_n\} \subset M$ with $u(x_n) \to 1$ it holds $|\nabla u(x_n)| \to 1$ the maximum principle at infinity is not valid. We emphasize that it is just the gradient condition that forces non-extensibility. Known examples show that there are geodesically complete model manifold without the Omori-Yau property but we do not know if the two concepts are independent. Apparently, the Omori-Yau principle is something stronger than the geodesic completeness. We also point out that, according to what we shall explain below, restating in our context a theorem of T. Lyons, [Ly], the (weak) Omori-Yau property is not invariant under quasi-isometries. It would be very interesting to give a direct proof of this fact using the maximum principle approach.
Using Theorem 8 of Section 1, we see that every differentiable manifold possesses a Riemannian metric with the Omori-Yau property. In the two dimensional case this is an immediate consequence of the uniformization theorem: accordingly, every Riemann surface has a geodesically complete (compatible) metric of constant curvature. As for higher dimensions, we can appeal, for instance, to a celebrated theorem by Lohkamp stating that there is no obstruction for a manifold to admit a complete metric with Ricci curvature pinched between negative

constants (see [Lo1]). However, in a sense, the validity of the Omori-Yau principle does not depend on curvature bounds as much as one would expenct (note that we have to control the laplacian of functions!). Indeed, according again to Theorem 8 above and quite surprisingly, the existence of a proper immersion with controlled mean curvature into some Euclidean space would suffice. Gauss equations then suggest that there exists even proper isometric minimal immersions with bad control of the curvature. This feeling is confirmed by the following striking example due to D. Stroock (see [St], page 133) showing that an exponential decay along diverging sequences is possible. Given a domain $D \subseteq \mathbb{R}^2$ and a conformal harmonic map $u = (u_1, ..., u_n) : D \to \mathbb{R}^n$ if we define $f : D \to \mathbb{R}^{n+2}$ by

$$f(x, y) = (x, y, u(x, y)) .$$

we get an imbedded, minimal surface $M = f(D)$ in $\mathbb{R}^{n+2}$. In case $D = \mathbb{R}^2$, this imbedding is also proper. The (induced) metric of M writes as

$$ds^2 = \left(1 + \left|\frac{\partial}{\partial x}u\right|^2\right)\left(dx^2 + dy^2\right)$$

where we have set $\frac{\partial}{\partial x}u = \left(\frac{\partial u_1}{\partial x}, ..., \frac{\partial u_n}{\partial x}\right)$, and its Gaussian curvature is given by the formula

$$K_{ds^2} = -\frac{\Delta_{\mathbb{R}^2} \log\left(1 + \left|\frac{\partial}{\partial x}u\right|^2\right)}{2\left(1 + \left|\frac{\partial}{\partial x}u\right|^2\right)}.$$

Suppose now that $n = 2$ and $u = u_1 + iu_2$ is holomorphic. Setting $z = x + iy$ and $u'(z), u''(z)$ for the (complex) derivatives of $u(z)$, the above expressions simplify to

$$ds^2 = \left(1 + |u'(z)|^2\right)|dz|^2 \text{ and } K_{ds^2}(z) = -2\frac{|u''(z)|^2}{\left(1 + |u'(z)|^2\right)^3}.$$

Following Stroock we chose $u(z)$ to be a primitive of $\sin e^z$ that is, the unique entire function satisfying $u(0) = 0$ and $u'(z) = \sin e^z$. Next, we consider the ray $\gamma(t) = t + i0$ and, having set $r(x) = dist_{ds^2}(0, x)$, we note that

$$r(\gamma(t)) \leq \int_0^t \sqrt{1 + (\sin e^t)^2}dt \leq 2t$$

whereas

$$K_{ds^2}(\gamma(t)) = -2\frac{\left(e^t \cos e^t\right)^2}{\left[1 + (\sin e^t)^2\right]^3} \leq -\frac{e^{2t}}{4}\left(\cos e^t\right)^2.$$

By taking $\{t_n\} = \{\log n\pi\}$ we thus see that, along the sequence $\{x_n\} = \{\gamma(t_n)\} \to \infty$,

$$K_{ds^2}(x_n) \leq -\frac{e^{r(x_n)}}{4}$$

as claimed. In order to better understand the role of curvatures we note that fast decays of the sectional(!) curvature are allowed only along (diverging) sequences.

Indeed, recasting in the Omori-Yau setting (see below) a result of Varopoulos concerning Brownian completeness, [Va], we have that the (weak) maximum principle at infinity does not hold on a geodesically complete Riemannian manifold $(M, \langle , \rangle)$ whose sectional curvature K satisfies $K(x) \leq -f(dist_M(x, o))$ for some origin $o \in M$ and some positive function $f \in \mathcal{C}^1([0, +\infty])$ such that $f' \leq cf^{\frac{3}{2}}$ on M and $f^{-\frac{1}{2}} \in L^1(+\infty)$. This result also shows that Theorem 8 is essentially sharp with respect to the curvature conditions expressed in Example 13.

Incidentally, we also observe that, because of the monotonicity formula, the existence of a (proper) minimal immersion into some $\mathbb{R}^n$ forces the manifold to have an at least polynomial volume growth. This kind of obstructions (in the large) however seems to disappear if we allow the mean curvature to increase. Therefore it is not so clear at this stage if every manifold (satisfying the Omori-Yau principle) can be properly immersed via controlled mean curvature in some Euclidean ambient.
The conformal version of the existence result for Omori-Yau metrics is also interesting but unknown. For instance one could try to prove that any Riemannian metric can be conformally deformed to a geodesically complete metric with bounded below Ricci tensor.

If we are interested in study global properties of $(M, \langle , \rangle)$ with respect to a selected family of stochastic processes, it is often (but not always) inessential that the manifold is "surely" non-extensible. By way of example, from the Brownian particle viewpoint a puncture in the plane is invisible. A stochastic puncture is surely more delicate! One could think of stochastic non-extensibility as the property that the processes lie with probability one in the state space, even if the state space is a piece of a bigger one.
Similarly to what happens in geometry, one considers a family of processes and say that M is complete with respect to this family if the (intrinsic) time-life (or explosion time) of each random path is almost-surely infinite. Said differently, any $[0, T]$-segment (we do not consider intervals $(0, T)$ because we are not interested in implosions) can be prolongated for all times.
Due to their known relations with geometry and analysis, very important concepts of stochastic completeness are obtained by taking Brownian motions or, more generally, the whole family of Γ-martingales on the underlying manifold. On the base of previous works by Darling and Zheng, [Da], [Zh], the study of martingale completeness was initiated by Emery in [Em1], see also the book [Em2], and we feel that it is strongly related to the maximum principle at infinity for the Hessian. The concept of Brownian completeness is earlier; a comprehensive account on the subject can be found in the nice survey paper of Grigor'yan, [Gr2].
We limit ourselves to consider Brownian completeness. The Brownian motion X_t on (the generic manifold) $(M, \langle , \rangle)$ starting from a point $x \in M$ is the Markovian, diffusion process whose transition density is given by the minimal heat kernel $p(x, y, t)$ of the Laplace operator Δ of M. According to Dodziuk, [Do], every

manifold possesses a unique minimal heat kernel so that the same holds for the Brownian particle. It happens that the Brownian motion X_t of M, or M for short, is (Brownian) complete if and only if for some (hence any) $(x,t) \in M \times (0,+\infty)$,

$$\int_M p(x,y,t)dy = 1. \tag{86}$$

Many other equivalent characterizations are available. In view of our purposes, the following is of special interest, see again [Gr2]: the Riemannian manifold $(M, \langle , \rangle)$ is Brownian complete if and only if

$$\begin{array}{c}\text{For every } \ \lambda > 0 \text{ the only non-negative,} \\ \text{smooth, bounded solution of } \Delta u = \lambda u \text{ on } M \text{ is } u \equiv 0.\end{array} \tag{87}$$

Here is the place where our next original contribution inserts. Using the above PDE aspect of Brownian completeness we are able to build a bridge between this concept and the maximum principles at infinity. More precisely, we have the following result from [PiRiSe1].

Theorem 40 *Let $(M, \langle , \rangle)$ be a (non necessarily geodesically complete) Riemannian manifold. Then the following are equivalent*
(a) M is Brownian complete
(b) For every $u \in \mathcal{C}^2(M)$ with $u^* = \sup_M u < +\infty$ and for every $\gamma > 0$,

$$\inf_{\Omega_\gamma} \Delta u \leq 0,$$

where we have set $\Omega_\gamma = \{x \in M : u(x) > u^* - \gamma\}$.
(c) For every $u \in \mathcal{C}^2(M)$ with $u^* = \sup_M u < +\infty$ there exists a sequence $\{x_k\}$ along which

$$u(x_k) > u^* - \frac{1}{k} \quad and \quad \Delta u(x_k) < \frac{1}{k}.$$

Proof. Since (b) $\Longrightarrow$ (c) is trivial and (c) $\Longrightarrow$ (a) follows from (87) it remains to prove (a) $\Longrightarrow$ (b). To this end, we reason by contradiction and assume that there is a function $u \in \mathcal{C}^2(M)$ with $u^* = \sup_M u < +\infty$, such that

$$\inf_{\Omega_\gamma} \Delta u \geq 2c$$

for some constant $c > 0$ and $\gamma > 0$. We let $\Omega^* = \{x \in M : \Delta u(x) > c\}$ so that $\Omega_\gamma \subset\subset \Omega^*$. Having set $\lambda = \frac{c}{\gamma}$ it is easy to see that $u - u^* + \gamma$ is a $\mathcal{C}^2$-subsolution of

$$\Delta u = \lambda u \tag{88}$$

on Ω^*. Since 0 is obviously a subsolution on M, we see that $u_\gamma = \max\{u - u^* + \gamma, 0\} \in \mathcal{C}^0(M) \cap H^1_{loc}(M)$ is a weak subsolution of (88) on all of M.

Furthermore, $u_\gamma \not\equiv 0$ and $0 \le u_\gamma \le \gamma$. Noting that any positive constant u_+ is a supersolution, choosing $u_+ > \gamma$ and applying the monotone iteration scheme (see [RaRiVe1]) yields a smooth solution v of (88) satisfying $u_\gamma \le v \le u_+$. Since u_γ does not vanish identically the same holds for v contradicting (87), hence Brownian completeness ■

As we claimed in the previous section, it turns out that the above conditions are equivalent to a further property which is related to a priori estimates of bounded solutions of some elliptic PDEs. Precisely:

(d) for any $f \in C^0(\mathbb{R})$ let $u \in C^2(M)$ satisfy $u^* = \sup_M u < +\infty$ and, for some $\gamma > 0$,

$$\Delta u \ge f(u) \text{ on } \Omega_\gamma \tag{89}$$

where we have set $\Omega_\gamma = \{x \in M : u(x) > u^* - \gamma\}$. Then $f(u^*) \le 0$.

Indeed, Theorem 20 of the previous section yields (c) $\Longrightarrow$ (d). On the other hand reasoning as above, we readily see that (d) $\Longrightarrow$ (b). Indeed, assume that condition (b) fail to hold. Then, we can construct a bounded above, $\mathcal{C}^2$-function on M satisfying the following properties: i) $0 < u^* < +\infty$; ii) u solves (89) with $f(t) = \lambda t$, for some constant $\lambda > 0$. This shows that (d) cannot hold.

In the previous section, we introduced the weak form of the maximum principle for the Laplace operator: accordingly, we say that the weak maximum principle at infinity holds on $(M, \langle , \rangle)$ if condition (c) is satisfied. We also showed its usefulness both in PDE and geometric problems. The chain of equivalence described above finally shows that this definition is in fact surprisingly deep in that it represents a kind of completeness. Observe that it does not imply (Riemannian) non-extensibility because the punctured plane satisfies the weak maximum principle and it is clearly extensible. Note however that, in a sense, the extended manifold cannot be arbitrarily large. For instance, given $(M, \langle , \rangle)$ satisfying the weak maximum principle, if $(N, (,))$ is geodesically complete (this assumption can be omitted using a different argument) and extends M in the Riemannian sense, then M is necessarily dense in N. To see this suppose that there exists some point o in the interior of $N \backslash M$ and let $d = dist_N(o, \partial M) = dist_N(o, p)$ for some $p \in \partial M$. We suppose ${}^N Ricci \ge -(m-1)c^2$ on $B_{2d}(o)$ for some constant $c > 0$; here $m = \dim M = \dim N$. Moving along a minimizing geodesic $\gamma \subset N \backslash M$ from o to p we reach a point o' such that $dist_N(o', p) = dist_N(o', \partial M) = d' < \frac{1}{2} inj\left(\overline{B_{2d}(o)}\right)$. Set $r(x) = dist_N(x, o')$ and define a radial smooth function on $B_{\frac{3}{2}d'}(o') \setminus \{o'\}$ by $u(x) = \coth^m(cr(x))$. Next, extend u smoothly on $N \setminus \{o'\}$ via a radial cut-off supported in $B_{2d'}(o')$. By restriction we thus obtain a smooth, bounded above function on M and, furthermore,

$$\sup_M u = u|_{\partial B_{d'}(o') \cap \partial M}.$$

On the other hand, a computation that uses the Laplacian comparison theorem yields

$$\Delta u > 0, \text{ on } \partial B_{d'}\left(o'\right)$$

provided o' has been chosen sufficiently near to p. This shows that the weak maximum principle cannot hold, as claimed. Needless to say, using (86) and Theorem 40, the same conclusion (in fact more), can be reached comparing the heat kernels of N and M via the parabolic maximum principle.
The validity of the weak maximum principle on geodesically complete manifolds is genuinely related to the volume growth of geodesic balls. The leading example is represented by model manifolds where a necessary and sufficient condition is that the function

$$\frac{vol\partial B_r}{volB_r} \notin L^1\left(+\infty\right) \tag{90}$$

(see Proposition 34 above). In the non-rotationally symmetric setting the best known result is due to Grigor'yan, [Gr1], who showed that

$$\frac{r}{\log volB_r} \notin L^1\left(+\infty\right)$$

suffices. It is an open conjecture that the weaker (90) is also sufficient on general manifolds.

The discovering of the conceptual meaning of the weak maximum principle displays several advantages: on one hand we have a new tool and a new approach to stochastic completeness. This can be appreciated recasting the results of the previous section in the stochastic contest (Kas'minskii test, Brownian completeness of manifolds which are isometric outside a compact set, characterizations on model manifolds etc...). In the other direction, one can deduce geometric and analytic results under very weak assumptions (see again the previous section); moreover the equivalence even suggests generalizations to new situations such as in the analysis of non-linear differential operators. Here is an example, a prelude of the topics of the next chapter.

Let $(M, \langle,\rangle)$ be a Riemannian manifold of dimension m. Given $p > 1$, the $p-$Laplacian $\Delta_p u$ of $u \in W^{1,p}\left(M\right)$ is defined by

$$\Delta_p u = div\left(\left|\nabla u\right|^{p-2}\nabla u\right).$$

In what follows we always assume $p \geq 2$. Similarly to the linear setting we say that $(M, \langle,\rangle)$ satisfies the weak p-maximum principle at infinity if for any $u \in \mathcal{C}^2\left(M\right)$ with $u^* = \sup_M u < +\infty$ there is a sequence $\{x_n\} \subset M$ along which

$$\begin{array}{ll} i) & u(x_n) > u^* - \frac{1}{n} \\ ii) & \Delta_p u(x_n) < \frac{1}{n}. \end{array}$$

Moreover, we have the following characterization which justifies its interest in PDE's theory.

Proposition 41 *The following are equivalent*

1. $(M, \langle,\rangle)$ *satisfies the weak p-maximum principle at infinity.*
2. *For each* $f \in \mathcal{C}^2(M)$ *and for each solution* $u \in \mathcal{C}^2(M)$ *of the problem*

$$\begin{cases} i) & \Delta_p u \geq f(u) \text{ on } M \\ ii) & u^* = \sup_M u < +\infty \end{cases}$$

it holds $f(u^*) \leq 0$.

Not every manifold satisfies the weak p-maximum principle at infinity, but when M is geodesically complete fairly sharp sufficient conditions are expressed in terms of the volume growth of geodesic balls, see [RiSe]. Specifying $p = 2$ and M to be a model manifold in the sense of Greene-Wu, a necessary and sufficient volume growth is given in (90). As for higher values of p, let us consider

$$(M, \langle,\rangle) = \left([0, +\infty) \times S^{m-1}, dr^2 + g(r)^2 d\theta^2\right)$$

where $d\theta^2$ denotes the standard metric of S^{m-1} and $g(r)$ is a smooth, positive function on $[0, +\infty)$, even at the origin and satisfying $g(0) = 0, g'(0) = 1$. Suppose furthermore that

$$\left(\frac{\int_0^r g(t)^{m-1} dt}{g(r)^{m-1}}\right)^{\frac{1}{p-1}} \in L^1(+\infty).$$

We recall that this latter can be written in the form

$$\left(\frac{volB_r}{vol\partial B_r}\right)^{\frac{1}{p-1}} \in L^1(+\infty)$$

where ∂B_r and B_r denote, respectively, the geodesic sphere and ball of radius r, centered at the pole of M. Consider the bounded above radial function

$$u(x) = \int_0^{r(x)} \left(\frac{volB_r}{vol\partial B_r}\right)^{\frac{1}{p-1}}.$$

Direct computations show $\Delta_p u \equiv const. > 0$ proving that $(M, \langle,\rangle)$ does not satisfies the weak p-maximum principle at infinity. Similarly to the linear case, we have

Theorem 42 *The model manifold* $(M, \langle,\rangle)$ *satisfies the weak p-maximum principle at infinity if and only if*

$$\left(\frac{volB_r}{vol\partial B_r}\right)^{\frac{1}{p-1}} \notin L^1(+\infty).$$

Proof. Necessity follows from the above discussion. Sufficiency is an immediate consequence of the next result. ■

Kahs'minskii, see [Kh], [Gr2], and Theorem 32 in the previous section, discovered a function theoretic property that guarantees the Brownian completentess of the manifold. The maximum principle approach to stochastic completeness introduced above suggests the following non linear extension.

Theorem 43 *Let $(M, \langle,\rangle)$ be any manifold. Assume that there exists a function $\gamma \in \mathcal{C}^2(M)$ such that*

$$\text{(91)} \qquad \begin{array}{ll} i) & \gamma(x) \to +\infty \text{ as } x \to \infty \\ ii) & \Delta_p \gamma \leq \lambda\gamma \text{ off a compact set,} \end{array}$$

for some $\lambda > 0$. Then, the weak p-maximum principle holds on M.

Proof. Since the case $p = 2$ is already known we assume $p > 2$. Without loss of generality we can suppose that

$$\begin{array}{ll} ii)' & \Delta_p \gamma \leq \gamma \text{ holds on } M \\ iii) & \gamma(x) > 0 \text{ on } M. \end{array}$$

Indeed, by adding a suitable constant to γ we readily obtain (91) *iii)* and .the validity of (91) *ii)* on all of M. In order to reduce our considerations to the case $\lambda = 1$ we use the positive-homogeneity of Δ_p. Precisely, if $a > 0$, then $\Delta_p(a\gamma) = a^{p-1}\Delta_p\gamma \leq a^{p-1}\lambda\gamma = \left(a^{p-2}\lambda\right)(a\gamma)$. The claim follows by choosing $a = \lambda^{-1/(p-2)}$.

Now, we let $u \in \mathcal{C}^2(M)$ be such that $u^* = \sup_M u < +\infty$. By contradiction, having defined

$$A_\eta = \{u > u^* - \eta\}, \; 0 < \eta < u^*$$

we assume

$$\Delta_p u \geq \eta \text{ on } A_\eta.$$

We fix $x_0 \in M$ satisfying

$$u(x_0) > u^* - \frac{\eta}{2}.$$

Next, we chose $0 < c \ll 1$ in such a way that

$$c\gamma(x_0) < \eta/2 - \varepsilon.$$

Thus

$$(u - c\gamma)(x_0) > u^* - \eta + \varepsilon.$$

Let Ω be the connected component of

$$\{x \in M : u(x) - c\gamma(x) > u^* - \eta + \varepsilon\}$$

containing x_0. Note that, on Ω,

$$\begin{aligned} c\gamma(x) &= (c\gamma(x) - u(x)) + u(x) \\ &< (u(x) - u^*) + \eta - \varepsilon \\ &< \eta - \varepsilon. \end{aligned}$$

Note also that

$$\Omega \subset\subset A_\eta$$

and

$$u - c\gamma = u^* - \eta + \varepsilon \text{ on } \partial\Omega.$$

Finally, on Ω,

$$\begin{aligned} \Delta_p u &\geq \eta > \eta - \varepsilon > c\gamma \\ &> c^{p-1}\gamma \geq \Delta_p (c\gamma + u^* - \eta + \varepsilon). \end{aligned}$$

It follows that

$$\begin{cases} \Delta_p u \geq \Delta_p (c\gamma + u^* - \eta + \varepsilon) & \text{on } \Omega \\ u = c\gamma + u^* - \eta + \varepsilon & \text{on } \partial\Omega \end{cases}$$

which implies (see Proposition 44 in the next section)

$$u - c\gamma \leq u^* - \eta + \varepsilon \text{ on } \Omega$$

a contradiction. ■

We conclude the section by comparing the proofs of Theorem 32 and Theorem 43: a key step that enables us to overcome the lack of linearity is to replace the use of the classical (point-wise!) maximum principle with a comparison on bounded domains. This method will be extensively used through the next chapter where we will obtain, among other things, a version of the Omori-Yau maximum principle at infinity for a large class of non-linear differential operators collected under the name of φ-Laplacians.

2 The nonlinear setting

This Chapter is devoted to some new aspects of the theory of partial differential inequalities involving a class of non-linear operators such as the p-Laplacian and the mean curvature operator.

In Section 1 we extend the validity of the Omori-Yau maximum principle at infinity to nonlinear situations under fairly sharp assumptions on the curvature of the manifold. The lack of linearity forces us to introduce a completely new approach to the problem.

In Section 2 we establish a number of comparison results on unbounded domains of a Riemannian manifold. Some of them, when specified for the mean curvature operator, extend on manifolds of any dimension previously known theorems for graphs in the Euclidean 3-space. Some others are new even in these situations. The technique is also new.

In Section 3 we deal with the mean curvature operator. We attempt to complete the L^∞-results of Section 2 exhibiting an asymptotic upper estimate.for exterior minimal graphs over (essentially) Cartan-Hadamard manifolds. To this end, we introduce an extended version of the classical comparison principle with catenoids due to Osserman.

2.1 The Omori-Yau maximum principle for the φ-Laplace operator under curvature assumptions

In Theorem 8 of Chapter 1, using the technique of Ahlfors-Yau, we proved a general version of the maximum principle at infinity for the Laplace-Beltrami operator. As a consequence, we obtained that this principle holds true provided the Ricci tensor is suitably bounded below. We already remarked that Yau technique makes an essential use of the linearity of the Laplacian, hence, this method cannot be implemented when dealing with non-linear differential operators such as, e.g., the mean curvature one.

In this section, we establish the validity of the maximum principle at infinity under curvature assumptions, for a large class of singular, elliptic and non-linear operators collected under the name of $\varphi-$Laplacians, see Corollary 52 below. The new approach we present here is based on a refined (finite) comparison argument and is somewhat related to a previous work of Redheffer, [Re]. The main results of the section, together with some others companion theorems, appear in [PiRiSe2].

From now on, until otherwise specified, $(M, \langle , \rangle)$ denotes a geodesically complete, non-compact, connected Riemannian manifold of dimension $\dim M = m$. To begin with, we introduce some terminology.
Let $\varphi \in \mathcal{C}^0([0, +\infty)) \cap \mathcal{C}^1((0, +\infty))$ be a real valued function satisfying the structural conditions

$$\text{(92)} \qquad \text{i) } \varphi(0) = 0; \quad \text{ii) } \varphi'(t) > 0, \, \forall t > 0; \quad \text{iii) } \varphi(t) \le At^\delta, \, \forall t \in [0, \varepsilon)$$

for some constants $A, \delta, \varepsilon > 0$ with possibly $\varepsilon = +\infty$. The $\varphi-$Laplacian of $u \in \mathcal{C}^1(M)$ is the divergence-form differential operator defined by

$$L_\varphi u = div\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right).$$

Of course, if the vector field in brackets is not $\mathcal{C}^1$, then the divergence must be understood in distributional sense. In this respect, note that the vector field may fail to be $\mathcal{C}^1$ at points where $\nabla u = 0$ even if u is assumed to be $\mathcal{C}^2$. As important natural examples we mention

1. the Laplace–Beltrami operator, Δu, corresponding to $\varphi(t) = t$;
2. or, more generally, the p-Laplacian, $div(|\nabla u|^{p-2}\nabla u)$, $p > 1$, corresponding to $\varphi(t) = t^{p-1}$;
3. the generalized mean curvature operator, $div\left(\frac{\nabla u}{(1+|\nabla u|^2)^\alpha}\right)$, $\alpha > 0$, corresponding to $\varphi(t) = t/(1+t^2)^\alpha$.

Under various assumptions on φ and in the Euclidean setting the φ-Laplacian has been studied by many authors over the past two decades. We mention in particular the seminal papers of J. Serrin and Serrin and L.A. Peletier [Se1], [PeSe], where Liouville type theorems are established. See also [Se2], [PuSeZo], [PuSe], where the maximum principle for the φ-Laplacian is investigated.

We say that a function $u \in \mathcal{C}^1(M)$ is φ-subharmonic if

$$div\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right) \geq 0 \text{ on } M. \tag{93}$$

Reversing the inequality, or replacing the inequality with an equality, we obtain the definition of φ-superharmonic and φ-harmonic function, respectively. In accordance to what we said above, when $|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u$ is not $\mathcal{C}^1$ the inequality in (93) means that

$$-\int_M \left\langle |\nabla u|^{-1} \varphi(|\nabla u|) \nabla u, \nabla\psi \right\rangle \geq 0$$

for all $0 \leq \psi \in \mathcal{C}_c^\infty(M)$ (non-negative compactly supported Lipschitz functions are enough). We mention that one of the basic problems is to determine sufficient geometric conditions so that (93) has only constant solutions. We thus naturally come into the world of φ-parabolic and φ-Liouville manifolds. The subject has been extensively studied in a recent paper of Rigoli and Setti, [RiSe].

It happens that φ-subharmonic functions satisfy a weak maximum principle on bounded domains. This is easily derived from the next comparison result due to Pucci, Serrin and Zou, [PuSeZo] (see also [RiSe]) which will play a decisive role in establishing the non-linear maximum principle at infinity.

Proposition 44 *Let $\Omega \subset\subset M$ be a pre-compact domain. Suppose that $u, v \in \mathcal{C}^1(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ satisfy*

$$(94)\quad \begin{array}{ll} i)\ div\left(|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u\right) \geq div\left(|\nabla v|^{-1}\varphi(|\nabla v|)\nabla v\right) & \text{weakly on } \Omega \\ ii)\ u \leq v & \text{on } \partial\Omega. \end{array}$$

Then $u \leq v$ on Ω.

Proof. Set $w = v - u$ in $\bar{\Omega}$ and, by contradiction, assume that there exists $x_1 \in \Omega$ such that $w(x_1) < 0$. Fix $\varepsilon > 0$ so small that $w(x_1) + \varepsilon < 0$. Since, by assumption, $w \geq 0$ on $\partial\Omega$, it follows that $w_\varepsilon = \min(w + \varepsilon, 0)$ is a non-positive, lipschitz function with compact support in Ω. By the distributional meaning of solutions of (94) i), taking $-w_\varepsilon$ as a test function, we get

$$(95)\qquad \int_\Omega h \leq 0$$

where we have set

$$h = \left\langle |\nabla v|^{-1}\varphi(|\nabla v|)\nabla v - |\nabla u|^{-1}\varphi(|\nabla u|)\nabla u, \nabla w_\varepsilon \right\rangle.$$

Clearly, $h = 0$ on the set $\{w + \varepsilon \geq 0\}$. On the other hand, on $\{w + \varepsilon < 0\}$,

$$\begin{aligned} h &= [\varphi(|\nabla v|) - \varphi(|\nabla u|)][|\nabla v| - |\nabla u|] \\ &\quad + \left[|\nabla v|^{-1}\varphi(|\nabla v|) + |\nabla u|^{-1}\varphi(|\nabla u|)\right][|\nabla u||\nabla v| - \langle\nabla u, \nabla v\rangle]. \end{aligned}$$

Whence, using Schwarz inequality and the monotonicity of φ, we obtain

$$h \geq 0 \quad \text{a.e. in } \Omega.$$

It follows from this and (95) that $h = 0$ a.e. in Ω, which in turn forces $\nabla u = \nabla v$ that is

$$\nabla w = 0 \quad \text{a.e. in } \Omega.$$

This shows that $w \equiv const.$ on Ω. Since $w(x_1) + \varepsilon < 0$ we deduce $w \leq -\varepsilon$ on Ω thus contradicting (94) ii). ■

Remark 45 The above proof works if $u, v \in Lip_{loc}(\Omega) \cap C^0(\overline{\Omega})$. These weaker regularity conditions are vital in many geometric situations, e.g., when we deal with radial function. Indeed it is known that, in general, radial functions are only locally Lipschitz on the underlying manifold.

Remark 46 We explicitly note that condition (92) iii) on φ is unnecessary for the result to hold

In case u is $\varphi-$subharmonic, we can take $v = \max_{\partial\Omega} u$ and conclude the validity of the weak maximum principle; namely, a $\varphi-$subharmonic function on a bounded domain Ω attains its maximum on the boundary $\partial\Omega$. The above proof shows that, in this case, φ is not required to be increasing.

Actually, $\varphi-$subharmonic functions satisfy also a strong maximum principle. For later purposes we recall (without proof) the following special case of Theorem 1 in [PuSe]. Again, condition (92) iii) is not needed.

Theorem 47 *Let $u \in \mathcal{C}^1(\Omega)$ be a $\varphi-$subharmonic function on the domain $\Omega \subset M$. If u achieves an interior maximum in Ω then u is constant on all of Ω.*

By means of Proposition 47 we readily obtain the following φ-version of the well-known "finite maximum principle" for the Laplace operator.

Corollary 48 *Let $\Omega \subseteq M$ be a domain and let $u \in \mathcal{C}^2(\Omega)$ be such that the vector field*

$$|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u \text{ is of class } \mathcal{C}^1 \text{ on } \Omega. \tag{96}$$

Then, at each point $x_0 \in \Omega$ of local maximum of u it holds

$$div\left(|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u\right)(x_0) \leq 0.$$

In particular, if $(N, \langle,\rangle)$ is a compact manifold without boundary and $u \in \mathcal{C}^2(N)$ satisfies the regularity condition (96) then, there exists $x_0 \in N$ such that

$$u(x_0) = \max_N u, \quad |\nabla u|(x_0) = 0, \quad div\left(|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u\right)(x_0) \leq 0.$$

Proof. Assume by contradiction that $div\left(|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u\right)(x_0) > 0$ where $x_0 \in \Omega$ is a local maximum for u. Then, because of (96), we find a small neighborhood $\tilde{\Omega} \subset\subset \Omega$ of x_0 such that, for each $x \in \tilde{\Omega}$, $u(x) \leq u(x_0)$ and

$$div\left(|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u\right)(x) > 0. \tag{97}$$

According to Theorem 47 $u(x) \equiv u(x_0)$ on $\tilde{\Omega}$ and, clearly, this contradicts (97). ■

Now we move from finite to infinity and we state the following general Theorem which represents the main result of the Section.

Theorem 49 *Let $(M, \langle,\rangle)$ be a complete manifold. Having fixed an origin $o \in M$, set $r(x) = dist_{(M,\langle,\rangle)}(x, o)$. Assume that the radial Ricci curvature of M satisfies*

$$Ricci(\nabla r, \nabla r) \geq -(m-1)B^2 G(r(x)) \quad \text{on } M \tag{98}$$

for some constant $B > 0$ and some positive, non-decreasing $G \in \mathcal{C}^1([0, +\infty))$.

Let $q \in \mathcal{C}^0([0,+\infty))$ *be a positive function such that*

$$(99) \qquad \begin{array}{l} i)\ \inf_{[0,+\infty)} q(t) > 0 \\ ii)\ q(t)\sqrt{G(t)} \leq z(t)^{\delta}, \quad t \gg 1, \end{array}$$

where $\delta > 0$ *is one of the structural constants of* φ *and* $z(t) \in \mathcal{C}^1(+\infty)$ *is positive, non-decreasing with*

$$iii)\ \frac{1}{z(t)} \notin L^1(+\infty).$$

Given $u \in \mathcal{C}^2(M)$ *with* $u^* = \sup_M u < +\infty$ *and such that the regularity condition (96) holds, define for* $0 < \eta < \min(u^*, \varepsilon)$

$$(100) \qquad A_\eta = \{x \in M : u(x) > u^* - \eta, \quad |\nabla u| < \eta\},$$

$\varepsilon > 0$ *being another structural constat of* φ*. Then*

$$(101) \qquad \inf_{A_\eta} \left[q(x) div \left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u \right)(x) \right] \leq 0.$$

Remark 50 As a matter of facts, the regularity assumption on the vector field $|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u$ can be omitted. Clearly, in this case, the conclusion of the Theorem must be modified accordingly, e.g. as follows. Fix any $0 < \eta \ll 1$ and $\sigma > 0$. Then $u \in C^1(M)$ is not a weak solution of the differential inequality

$$q(r(x)) div \left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u \right) \geq \sigma > 0, \quad on\ A_\eta.$$

Remark 51 Using a by-contradiction argument, it can be shown that the latter version of Theorem 49 is equivalent to the following statement.

Suppose $(M, \langle,\rangle), q, G, z$ as in Theorem 49. Let $u \in C^1(M)$ be a weak solution of

$$\begin{cases} div \left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u \right) \geq b(x) f(u), & on\ A_\eta \\ u^* = \sup_M u < +\infty, \end{cases}$$

for some $0 < \eta < u^*$, where $f, b \in C^0(\mathbb{R})$ satisfy $b(x) \geq 0$ on M and $b(x) \geq 1/q(r(x))$ whenever $r(x) \gg 1$. Then $f(u^*) \leq 0$.

The φ-version of the usual Omori-Yau maximum principle announced above follows by rephrasing Theorem 49 with $q(t) \equiv 1$.

Corollary 52 *Assume that the curvature condition (98) holds and that the non-decreasing function* G *satisfies*

$$\sqrt{G(t)} \leq z(t)^{\delta}$$

with $z(t), \delta$ as in Theorem 49. Let $u \in \mathcal{C}^2(M)$ be such that $u^* = \sup_M u < +\infty$ and assume that the vector field $|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u$ is of class $\mathcal{C}^1$ on M. Then there exists a sequence $\{x_k\} \subset M$ with the following properties

i) $u(x_k) > u^* - \frac{1}{k}$
ii) $|\nabla u|(x_k) < \frac{1}{k}$
iii) $div\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right)(x_k) < \frac{1}{k}$

for each $k \in \mathbb{N}$.

Proof (of Theorem 49). In all that follows, B_r and ∂B_r represent the geodesic ball and sphere, respectively, centered at o and of radius $r > 0$.

We reason by contradiction and suppose that

$$q(x) div\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right)(x) \geq \sigma_0 > 0, \text{ on } A_\eta \tag{102}$$

for some η. The idea of the proof is to show that one can construct a suitable radial function v such that
(a) $u - v$ attains a positive maximum m on a bounded domain $\Omega \subset\subset M$.
(b) $u - v < m$ on $\partial\Omega$.
(c) $div\left(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u\right) \geq div\left(|\nabla v|^{-1} \varphi(|\nabla v|) \nabla v\right)$ on Ω, thereby contradicting the weak comparison principle of Proposition 44.

First, we observe that u^* cannot be achieved at any point x_0 of M. For, otherwise, such a x_0 would belong to the open set A_η where (102) is satisfied and this would contradict Corollary 48.

Since u^* is not attained on M, we find a divergent sequence $\{r_j\}_{j \in \mathbb{N}}$ such that

$$\sup_{\partial B_{r_j}} u(x) \to u^*, \text{ as } j \to +\infty. \tag{103}$$

Choose $R_2 > 0$ in such a way that

$$u^*_{R_2} := \sup_{B_{R_2}} u > u^* - \eta.$$

Next, fix $\alpha_{R_2} \in \left(u^*_{R_2}, u^*\right)$ and, for every $\sigma \in (0, \sigma_0)$ sufficiently small, define a function $\alpha_\sigma(t)$ by the formula

$$\alpha_\sigma(t) = \alpha_{R_2} + \int_{R_2}^t \varphi^{-1}\left(\frac{\sigma \int_{R_2}^s \frac{g(y)^{m-1}}{q(y)} dy}{g(s)^{m-1}}\right) ds \tag{104}$$

where we have set

$$g(y) = \frac{1}{D\sqrt{G(0)}}\left\{e^{D\int_0^y \sqrt{G(s)}ds} - 1\right\}, \forall y \geq 0,$$

$D > 0$ a suitable constant.

We collect in the following lemma the key properties of α_σ.

Lemma 53 *If $\sigma > 0$ is sufficiently small, then α_σ is defined on $[R_2, +\infty)$, is non-decreasing and satisfies*

$$\left\{ \begin{array}{l} q\left(g^{m-1}\varphi\left(\alpha_\sigma'\right)\right)' = \sigma g^{m-1} \\ \alpha_\sigma'(R_2) = 0, \quad \alpha_\sigma(R_2) = \alpha_{R_2} \end{array} \right. \tag{105}$$

and

$$\lim_{t\to+\infty} \alpha_\sigma(t) = +\infty. \tag{106}$$

Moreover, the following hold:
(107)
(a) $\alpha_\sigma(t) \to 0^+$ as $\sigma \to 0^+$ uniformly on compact subsets of $[R_2, +\infty)$
(b) $\alpha_\sigma'(t) \to 0^+$ as $\sigma \to 0^+$ uniformly on $[R_2, +\infty)$.

Postponing the proof of the lemma, we continue with the proof of the theorem. To better illustrate the main idea, we first assume that $o \in M$ is a pole so that the distance function $r(x)$ is smooth on $M\setminus\{o\}$.

We define a function v_σ on $M\setminus B_{R_2}$ by the formula

$$v_\sigma(x) = \alpha_\sigma(r(x)).$$

Then, according to (106)

$$v_\sigma(x) \to +\infty, \quad \text{as } r(x) \to +\infty. \tag{108}$$

Furthermore, an application of Laplacian comparison Theorem, see [GrWu], yields that, up to choosing $D > 0$ large enough,

$$\Delta r \leq (m-1)\frac{g'}{g}(r(x)), \quad \text{on } M\setminus\{o\}.$$

Thus, a computation that uses (105) gives

$$q\ div\left(|\nabla v_\sigma|^{-1}\varphi(|\nabla v_\sigma|)\nabla v_\sigma\right) \leq \sigma_0, \quad \text{on } M\setminus\overline{B_{R_2}}. \tag{109}$$

We claim that, if σ is sufficiently small, then $u - v_\sigma$ attains a positive maximum m_σ on $M\setminus\overline{B_{R_2}}$. Indeed by (103) we may choose $\bar{j}$ sufficiently large that, having set $R_3 = r_{\bar{j}}$, we have

$$R_3 > R_2 \quad \text{and} \quad \sup_{\partial B_{R_3}} u > \alpha_{R_2}.$$

We select $\bar{\eta} > 0$ small enough that $\alpha_{R_2} + \bar{\eta} < \sup_{\partial B_{R_3}} u$. Finally, according to (107) (a), we choose $\sigma = \sigma(R_3, \bar{\eta}) \in (0, \sigma_0)$ so small that

$$\alpha_{R_2} \leq \alpha_\sigma(t) \leq \alpha_{R_2} + \bar{\eta}, \quad [R_2, R_3].$$

For every such σ we have

$$v_\sigma(x) \geq \alpha_\sigma(R_2) = \alpha_{R_2} > \sup_{\overline{B_{R_2}}} u \geq \sup_{\partial B_{R_2}} u \geq u(x), \quad \forall x \in \partial B_{R_2}$$

so that

$$u - v_\sigma < 0, \quad \text{on } \partial B_{R_2}.$$

Furthermore, if $\bar{x} \in \partial B_{R_3}$ is such that $\sup_{\partial B_{R_3}} u = u(\bar{x})$ then

$$(u - v_\sigma)(\bar{x}) = \sup_{\partial B_{R_3}} u - \alpha_\sigma(R_3) \geq \sup_{\partial B_{R_3}} u - \alpha_{R_2} - \bar{\eta} > 0.$$

Finally, (108) and the fact that $u^* < +\infty$ imply that

$$(u - v_\sigma)(x) < 0, \quad \text{for } r(x) \gg 1.$$

Thus, $u - v_\sigma$ achieves its absolute, positive maximum m_σ on $M \backslash \overline{B_{R_2}}$ proving the claim. Moreover, the set

$$\Gamma_\sigma = \{x \in M : (u - v_\sigma)(x) = m_\sigma\}$$

is compact and contained in $M \backslash \overline{B_{R_2}}$.

Our next purpose is to show that, up to choosing σ small enough,

$$\Gamma_\sigma \subset A_\eta. \tag{110}$$

To this end, we first note that, for each $\nu > 0$, there exists $\sigma_1 = \sigma_1(\nu) > 0$ such that, whenever $0 < \sigma < \sigma_1$, we have

$$v_\sigma(\xi) < \nu + \alpha_{R_2}, \quad \forall \xi \in \Gamma_\sigma.$$

Indeed, $v_\sigma(x) = \alpha_\sigma(r(x))$ and $r(\Gamma_\sigma) \subset (R_2, +\infty)$ is compact, therefore we can use property (107) (a) of α_σ. Next, we observe that, as a consequence of (107) (b) and Gauss Lemma, for each $\nu > 0$ there exists $\sigma_2 = \sigma_2(\nu) > 0$ such that, if $0 < \sigma < \sigma_2$,

$$|\nabla v_\sigma|(\xi) = \alpha'_\sigma(\xi) < \nu, \quad \forall \xi \in \Gamma_\sigma.$$

We may therefore choose σ sufficiently small that

$$u > u^* - \frac{\eta}{2} \quad \text{and } |\nabla v_\sigma| < \frac{\eta}{2} \text{ on } \Gamma_\sigma;$$

since, by definition of Γ_σ,

$$|\nabla v|(\xi) = |\nabla v_\sigma|(\xi), \quad \forall \xi \in \Gamma_\sigma$$

we conclude the validity of (110). In particular, recalling that A_η is open and Γ_σ compact, we find a small open neighborhood of Γ_σ which is contained in A_η.

Now, pick a point $\xi \in \Gamma_\sigma$, fix any $0 < \mu < m_\sigma$ and set $\Omega_{\mu,\xi}$ for the connected component containing ξ of the set

$$\left\{x \in M \backslash \overline{B_{R_2}} : (u - v_\sigma)(x) > \mu\right\}.$$

Clearly $\Omega_{\mu,\xi}$ is bounded, $\xi \in \Omega_{\mu,\xi}$ and, since $u - v_\sigma < 0$ on ∂B_{R_2}, $\overline{\Omega}_{\mu,\xi} \subset M \backslash \overline{B_{R_2}}$. Furthermore, $u = v_\sigma + \mu$ on $\partial\Omega_{\mu,\xi}$ and

$$u(x) > v_\sigma(x) + \mu \geq \alpha_{R_2} > \sup_{B_{R_2}} u > u^* - \eta \quad \text{on } \Omega_{\mu,\xi}.$$

Therefore, because of (110), using also Proposition 5 of Chapter 1, we can choose μ sufficiently near to m_σ in such a way that

$$\overline{\Omega}_{\mu,\xi} \subset A_\eta.$$

But, according to (109) and (102) we have on A_η

$$div\left(|\nabla u|^{-1}\varphi\left(|\nabla u|\right)\nabla u\right) \geq div\left(|\nabla\left(v_\sigma+\mu\right)|^{-1}\varphi\left(|\nabla\left(v_\sigma+\mu\right)|\right)|\nabla\left(v_\sigma+\mu\right)|\right).$$

Thus, applying the comparison principle of Proposition 44 we deduce

$$u \leq v_\sigma + \mu, \quad \text{on } \Omega_{\mu,\xi}$$

contradicting the definition of ξ.

We now drop the assumption the o is a pole of M and describe how to adapt the argument to deal with the lack of smoothness of the distance function $r(x)$ on the cut locus $cut(o)$.

The reasonings given above may be carried out without changes to deduce that there exists a sufficiently small σ such that $(u - v_\sigma)$ attains its maximum on the compact set $\Gamma_\sigma \subset M\backslash\overline{B_{R_2}}$. Moreover,

$$u(\xi) > u^* - \frac{\eta}{2} \quad \text{and } 0 < \alpha'_\sigma\left(r\left(\xi\right)\right) < \frac{\eta}{2}, \tag{111}$$

for every $\xi \in \Gamma_\sigma$. From these ingredients, as before, we would like to conclude that $|\nabla u\left(\xi\right)| < \eta/2$ on Γ_σ. Since, by direct computations, (109) holds weakly on $M\backslash\overline{B_{R_2}}$, the previous comparison argument can be equally used to reach a contradiction. The problem comes from the fact that we can no longer assume that v_σ is smooth and, in particular, that $|v_\sigma| < \eta/2$ on Γ_σ. To circumvent the problem, we adopt a trick of Calabi as follows. Let $\xi \in \Gamma_\sigma \cap cut(o)$ and let γ be a minimizing geodesic, parametrized by arc-length, joining o to ξ. For $\varepsilon > 0$ suitably small, denote by $o_\varepsilon = \gamma(\varepsilon)$ and by $r_\varepsilon(x)$ the distance function from o_ε. Thus $\xi \notin cut(o_\varepsilon)$ and r_ε is smooth in a neighborhood of ξ. We define

$$\bar{v}_\sigma(x) = \alpha_\sigma\left(r_\varepsilon(x) + \varepsilon\right).$$

By the triangle inequality and the monotonicity of α_σ, it holds $\bar{v}_\sigma \geq v_\sigma$; therefore

$$u - \bar{v}_\sigma \leq u - v_\sigma \quad \text{on } M\backslash\overline{B_{R_2}}. \tag{112}$$

On the other hand, along the piece of γ between o_ε and ξ it holds $\bar{v}_\sigma = v_\sigma$ so that

$$\left(u - \bar{v}_\sigma\right)\circ\gamma\left(t\right) = \left(u - v_\sigma\right)\circ\gamma(t) \quad \forall t \in [\varepsilon, l\left(\gamma\right)]. \tag{113}$$

Furthermore, along the same segment, but near ξ where $r_\varepsilon(x)$ is smooth, $|\nabla\bar{v}_\sigma| = \alpha'_\sigma\left(\varepsilon + r_\varepsilon\right) = \alpha'_\sigma\left(r\right)$. In particular,

$$|\nabla\bar{v}_\sigma|\left(\xi\right) < \frac{\eta}{2}. \tag{114}$$

Now, from (112) and (113) it follows that the maximum of $u - \bar{v}_\sigma$ is m_σ and is attained at ξ. Since $u - \bar{v}_\sigma$ is of class $\mathcal{C}^1$ near ξ we have

$$|\nabla u|(\xi) = |\nabla \bar{v}_\sigma|(\xi).$$

This latter together with (114) gives $|\nabla u|(\xi) < \eta/2$ as desired. It remains to prove Lemma 53. ∎

Proof (of Lemma 53). We set, for each $t \geq R_2$,

$$S(t) = \frac{\int_{R_2}^{t} \frac{g(y)^{m-1}}{q(y)} dy}{g(t)^{m-1}}$$

and we note that

$$0 \leq S(t) \leq \frac{1}{\inf q} \frac{\int_{R_2}^{t} g(y)^{m-1} dy}{g(t)^{m-1}} \leq \frac{C}{\sqrt{G(t)}},$$

for some constant $C > 0$. Therefore,

$$0 < \sup_{[R_2, +\infty)} S(t) = S^* < +\infty$$

and we can choose $\bar{\sigma} > 0$ in such a way that, for each $0 < \sigma < \bar{\sigma}$

$$\sigma S(t) \in Dom\left(\varphi^{-1}\right), \quad \forall t \geq R_2.$$

This proves that, for such values of σ, α_σ is a well defined function on $[R_2, +\infty)$. Clearly, by definition, α_σ is non-decreasing and satisfies (105). In order to prove (106) it suffices to show that

$$\varphi^{-1}(S(t)) \geq \frac{C}{z(t)}, \quad t \gg 1,$$

for some constant $C > 0$. This follows if there exists $C > 0$ such that

$$\tilde{h} = \frac{S(t)}{\varphi\left(\frac{C}{z(t)}\right)} \geq \frac{1}{\sigma}, \quad t \gg 1. \tag{115}$$

We shall show that (115) holds provided

$$0 < C < \left[\frac{\sigma}{DA(m-1)}\right]^{1/\delta}. \tag{116}$$

Without loss of generality, we may assume that $z(t) \to +\infty$ as $t \to +\infty$; thus, the structural condition (92) iii) of φ gives

$$\tilde{h}(t) \geq \frac{A(t)}{B(t)}$$

with

$$A(t) = z(t)^{\delta} \int_{R_2}^{t} g(y)^{m-1} q(y) dy, \quad B(t) = AC^{\delta} g(t)^{m-1}.$$

Now, since $A(t) \to +\infty$ as $t \to +\infty$, (115) is trivially met if $B(t)$ is bounded. Otherwise,

$$\liminf_{t \to +\infty} \frac{A(t)}{B(t)} \geq \liminf_{t \to +\infty} \frac{A'(t)}{B'(t)}.$$

A computation that uses $z' \geq 0, q > 0$ and $G' \geq 0$ shows that

$$\frac{A'(t)}{B'(t)} \geq \frac{1}{AC^{\delta}(m-1)} \frac{\sqrt{G(t)}}{\frac{g'(t)}{g(t)}}, \quad t \gg 1,$$

and since $g'(t)/g(t) \geq D\sqrt{G(t)}$ the condition imposed on C implies that

$$\liminf_{t \to +\infty} \frac{A'(t)}{B'(t)} > \frac{1}{\sigma}$$

so that (115) holds even in this case. This proves (106).

Observe that (107) (a) is an immediate consequence of (107) (b); hence it remains to prove this latter. By the monotonicity of φ^{-1} it holds

$$\alpha'_{\sigma}(t) = \varphi^{-1}(\sigma S(t)) \leq \varphi^{-1}(\sigma S^{*})$$

and since $\lim_{t \to 0^{+}} \varphi^{-1}(t) = \varphi^{-1}(0) = 0$ it follows that

$$\sup_{[R_2, +\infty)} \alpha'_{\sigma}(t) \leq \sup_{[R_2, +\infty)} \varphi^{-1}(\sigma S^{*}) \to 0 \text{ as } \sigma \to 0^{+},$$

completing the proof. ■

2.2 Comparison principles at infinity for non-linear differential operators

(A) Introduction and purposes. Comparison principles are invaluable tools both in the analytic and geometric theory of PDEs on Riemannian manifolds. Clearly, they are mainly used to establish uniqueness of certain objects. Local theories are not sensitive of the geometry of the domain manifold and depend only on the nature of the operators involved. In contrast, when we deal with global theories, and comparisons at infinity lie in this category, the geometry often plays a decisive role. In the linear setting, since we can always compare with a fixed object (typically a null solution) we are merely reduced to Liouville theories. This is the reason why we concentrate on non-linear situations.

Comparison results at infinity for the mean curvature operator have a long tradition starting from the pioneering works of Bers, see the book [Os], and Langevin-Rosenberg [LaRo] in minimal surface theory.

Theorem 54 (Langevin-Rosenberg) *Let $\Omega \subset \mathbb{R}^2$ be an exterior domain with smooth boundary $\partial\Omega$ and let $u_1, u_2 \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0\left(\bar{\Omega}\right)$ be two solutions of the exterior Plateau problem*

$$\begin{cases} div\left(\frac{\nabla u_1}{\sqrt{1+|\nabla u_1|^2}}\right) = 0 = div\left(\frac{\nabla u_2}{\sqrt{1+|\nabla u_2|^2}}\right), & \text{on } \Omega \\ u_1 = u_2, & \text{on } \partial\Omega. \end{cases}$$

If $u_1 \leq u_2$ on Ω and $\liminf_{|x|\to+\infty}(u_2(x) - u_1(x)) = 0$ then $u_1 = u_2$, i.e., the minimal graphs Γ_{u_1} and Γ_{u_2} associated to u_1 and u_2 coincide.

We point out that in the situation pictured in Theorem 54 the minimal graphs are necessarily asymptotic at infinity, i.e., $u_2(x) - u_1(x) = o(1)$ as $|x| \to +\infty$. Indeed, both Γ_{u_1} and Γ_{u_2} have limiting normals at infinity and these must coincide since Γ_{u_1} lies below Γ_{u_2}. Without loss of generality we can suppose that the common limiting normal is $N = (0, 0, 1)$. Thus, the asymptotic expansions of u_1 and u_2 near infinity are given by, see e.g. [Sc] and [LaRo],

$$u_i(x) = a_i \log|x| + b_i + o(1) \text{ as } |x| \to +\infty, \quad i = 1, 2$$

for some constants $a_i, b_i \in \mathbb{R}$. Since $\liminf_{|x|\to+\infty}(u_2(x) - u_1(x)) = 0$ we conclude that $a_1 = a_2$ and $b_1 = b_2$ as claimed.

From the geometric viewpoint Theorem 54 asserts that if two properly imbedded minimal surfaces of finite total curvature and compact boundary do not intersect then they are a positive distance apart. In this direction the result was developed and generalized by Meeks and Rosenberg, [MeRo], to compactly bordered minimal surfaces properly immersed in complete, flat 3-manifolds. The case where the surfaces have empty boundary was previously obtained by Hoffman and Meeks in [HoMe]: this beautiful comparison at infinity, known as the Strong Half-Space theorem, states that if two properly immersed minimal surfaces in $\mathbb{R}^3$ do not intersect then they must be parallel planes.

In the analytic direction Langevin-Rosenberg Theorem was extended in various ways to obtain uniqueness results for prescribed mean curvature graphs over unbounded domains of $\mathbb{R}^2$. In view of our purposes, we limit ourselves to quote the papers of Miklyukov, [Mi], Collin and Krust, [CoKr], and Hwang [Hw], [Hw1]. They are joint by the common idea of using a deep structural inequality of the mean curvature operator, see below. Here is a far reaching version of their result

Theorem 55 (Hwang) *Let $\Omega \subset \mathbb{R}^2$ be an unbounded domain and let $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ be a solution of the problem*

$$\begin{cases} div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = div\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right), & on\ \Omega \\ u = v, & on\ \partial\Omega. \end{cases}$$

Set ∂B_r for the Euclidean circle centered at the origin of radius $r > 0$ and suppose that

$$\sup_{\Omega\cap\partial B_r} |u - v| = o\left(\int^r \frac{ds}{vol\,(\partial B_s \cap \Omega)}\right),\ as\ r \to +\infty. \tag{117}$$

Then $u \equiv v$ on Ω.

We remark the role of (117). Rephrasing the theorem, uniqueness is guaranteed provided the distance between the graphs does not growth too fast with respect to the "volume growth" of $\Omega \cap \partial B_r$. In this sense, Theorem 55 can be considered as an L^∞-comparison principle at infinity.

By way of example, this gives uniqueness of minimal, or constant mean curvature graphs over a strip such as the helicoid (minimal surface) or the Delunay surface (constant mean curvature). As a matter of facts both analytic and geometric uniqueness theorems for Delunay type surfaces (including a comparison principle at infinity and higher dimensional generalizations) was earlier obtained by Sa Earp and Rosenberg in [EaRo].

Following [PiRiSe], this section aims to prove some L^q-comparison principles at infinity on complete manifolds for a class of non-linear differential operators. When specialized to the mean curvature operator, and for $q = \infty$, our results extend to Riemannian manifolds the above mentioned theorems; the cases $q < +\infty$ are new even in the Euclidean setting. On the other hand, the generality of our arguments enables us to deal with the case of the p-Laplacian, $p \geq 2$, and to exhibit uniqueness theorems on unbounded domains even in this situation. The key ingredients are represented by Lemmas 60, 63 below.

In all that follows $(M, \langle,\rangle)$ always denote a complete, non compact, connected Riemannian manifold of dimension $m = \dim M$. Moreover, having fixed an origin $o \in M$, we set $r(x) = dist_M(o, x)$ and we denote by $\partial B_r, B_r$ the geodesic sphere and ball centered at o of radius $r > 0$.

(B) The operators. Let $\varphi \in \mathcal{C}^1((0,+\infty)) \cap \mathcal{C}^0([0,+\infty))$ satisfy

(118) $\quad i)\ \varphi(0)=0; \qquad ii)\ \varphi(t)>0$ if $t>0; \qquad iii)\ \varphi(t)$ increasing if $t>0$.

Moreover, assume that there exist a positive functional Ψ and constants $0 < C_1, C_2$, $0 < B \leq +\infty$ and $0 < b < a$ such that, for every continuous vector fields X, Y with $|X|, |Y| < B$, it holds

$$
(??)\qquad
\begin{array}{ll}
iii) & p \mapsto \Psi(X_p, Y_p) \text{ is continuous} \\
iv) & \Psi(X,Y) = \Psi(Y,X) \\
v) & \Psi(X,Y) \geq 0 \text{ equality holding at } p \text{ iff } X_p = Y_p. \\
vi) & \left\langle |X|^{-1}\varphi(|X|)X - |Y|^{-1}\varphi(|Y|)Y\ ,\ X-Y \right\rangle \geq C_1 \Psi(X,Y)^a \\
vii) & \left| |X|^{-1}\varphi(|X|)X - |Y|^{-1}\varphi(|Y|)Y \right| \leq C_2 \Psi(X,Y)^b.
\end{array}
$$

We shall consider differential operators defined for $u \in \mathcal{C}^2(M)$ by

$$
div\left(|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u\right) \tag{119}
$$

which we already called the φ-Laplacian of u. Without further mention, from now on we always assume that the operators (119) satisfy the structural conditions (118).

We recall from the previous section that, under the solely assumptions (118) $i), ii), iii)$, the φ-Laplacian satisfies the (weak) comparison principle on precompact domains of M. This will be tacitly used when necessary.

The following are the prototypes of our operators.

1. Let $\varphi(t) = t/\sqrt{1+t^2}$ so that (119) becomes $div(\nabla u/\sqrt{1+|\nabla u|^2})$ the mean curvature operator. We have

Lemma 56 (Miklyukov-Hwang-Collin-Krust inequality) *Let $(V, \langle,\rangle)$ be a real vector space endowed with an inner product. Then, for every $x, y \in V$,*

$$
\left\langle \frac{x}{\sqrt{1+|x|^2}} - \frac{y}{\sqrt{1+|y|^2}}, x-y \right\rangle \geq \frac{\sqrt{1+|x|^2}+\sqrt{1+|y|^2}}{2} \left| \frac{x}{\sqrt{1+|x|^2}} - \frac{y}{\sqrt{1+|y|^2}} \right|^2 \tag{120}
$$

equality holding if and only if $|x| = |y|$.

Proof. Set for brevity $\mathcal{M}(x) = x/\sqrt{1+|x|^2}$. Direct computations show that

$$
\begin{aligned}
\langle \mathcal{M}(x) - \mathcal{M}(y), x-y \rangle &= \frac{\sqrt{1+|x|^2}-\sqrt{1+|y|^2}}{2}\left(|\mathcal{M}(x)|^2 - |\mathcal{M}(y)|^2\right) \\
&+ \frac{\sqrt{1+|x|^2}+\sqrt{1+|y|^2}}{2}|\mathcal{M}(x)-\mathcal{M}(y)|^2.
\end{aligned}
$$

To conclude, note that by the strict monotonicity of $\varphi(t) = t/\sqrt{1+t^2}$, it holds

$$
\frac{\sqrt{1+|x|^2}-\sqrt{1+|y|^2}}{2}\left(|\mathcal{M}(x)|^2 - |\mathcal{M}(y)|^2\right) \geq 0
$$

equality holding if and only if $|x| = |y|$. ■

As a consequence, we see that the structural conditions (118) are met with

$$\begin{array}{l} \Psi(X,Y) = \left| \frac{X}{\sqrt{1+|X|^2}} - \frac{Y}{\sqrt{1+|Y|^2}} \right| \\ C_1 = C_2 = 1 \\ a = 2,\ b = 1 \\ B = +\infty. \end{array}$$

2. Let $\varphi(t) = t^{p-1}$ so that (119) reads $div\left(|\nabla u|^{p-2}\nabla u\right)$, the p-Laplace operator. The following inequality is due to Lindqvist, [Li].

Lemma 57 *Let $(V, \langle , \rangle)$ be a real vector space endowed with an inner product. If $p \geq 2$ then, for every $x, y \in V$,*

$$|x|^p + (p-1)|y|^p \geq p|y|^{p-2}\langle x, y\rangle + \frac{1}{2^{p-1}-1}|x-y|^p. \tag{121}$$

Corollary 58 *In the assumptions and notations of Lemma 57, for every $x, y \in V$, it holds*

$$\left\langle |x|^{p-2}x - |y|^{p-2}y, x-y \right\rangle \geq \frac{2}{p\left(2^{p-1}-1\right)}|x-y|^p. \tag{122}$$

Proof. Set for brevity $A(x) = |x|^{p-2}x$ and note that

$$\langle A(x) - A(y), x-y\rangle = |x|^p + |y|^p - \langle x, y\rangle\left(|x|^{p-2} + |y|^{p-2}\right). \tag{123}$$

On the other hand, applying (121) twice with the role of x and y interchanged we get

$$p\left(|x|^p + |y|^p\right) \geq p\left(|x|^{p-2} + |y|^{p-2}\right)\langle x, y\rangle + \frac{2}{2^{p-1}-1}|x-y|^p.$$

Inserting this latter into (123) yields the validity of (122). ■

We note that, when $p \geq 2$, the application $x \mapsto A(x)$ is locally lipschitz. In fact, we have

Lemma 59 *In the above assumptions and notations, having fixed $B > 0$, there is a constant $C = B^{p-2}\left(1 + 2^{p-1}\right) > 0$, such that for each $x, y \in V$ with $|x|, |y| < B$ it holds*

$$\left| |x|^{p-2}x - |y|^{p-2}y \right| \leq C|x-y|. \tag{124}$$

Proof. Fix $|x|, |y| \leq B$ and assume that, for $0 \leq t \leq 1$, $tx + (1-t)y \neq 0$. Then, direct computations show

$$\left|\frac{d}{dt}A(tx + (1-t)y)\right| \leq (p-1)|tx + (1-t)y|^{p-2}|x-y|.$$

It follows

$$\begin{aligned}|A(x)-A(y)| &= \left|\int_0^1 \frac{d}{dt}A(tx+(1-t)y)\right| \\ &\leq c_1 |x-y|\end{aligned}$$

where we have set $c_1 = (p-1)\, B^{p-2}$.

The case $tx+(1-t)y = 0$ for some $0 \leq t \leq 1$ can be checked directly. Indeed if $t = 0, 1$ then $|A(x)| = |x|^{p-1}$ and the desired inequality holds with $c_2 = B^{p-2}$. Otherwise, assume that $1/2 \leq t < 1$ (the other case being similar). Since $x = y\,(t-1)/t$ we conclude

$$\begin{aligned}|A(x)-A(y)| &\leq \left(\frac{|t-1|^{p-1}}{t^{p-1}}+1\right)|y|^{p-2}|y| \\ &\leq \left(1+2^{p-1}\right)B^{p-2}t\,|x-y| \leq \left(1+2^{p-1}\right)B^{p-2}\,|x-y|\,.\end{aligned}$$

■

Warning. *As the above proof shows, if $tx+(1-t)y = 0$ for some t, both inequality (124) and the restrictions $p \geq 2$, $|x|, |y| < B$ are quite unnatural. In fact, in these cases (and only in these cases) we could always write $|A(x)-A(y)| \leq C\,|x-y|^{p-1}$ without any further assumption. Due to this fact, some of the below comparisons for the p-Laplacian could appear quite strange, e.g., if one of the functions vanishes identically (p-parabolicity). In order to overcome these problems we could simply correct ad-hoc the structural conditions (118) on the φ-Laplace operator. In any case, since we are mainly interest in showing a technique, for the clarity of exposition we prefer to maintain the present status of generality.*

Summarizing, the structural conditions (118) are satisfied provided

$$\begin{aligned}&\Psi(X,Y) = |X-Y| \\ &B > 0 \\ &C_1 = \tfrac{2}{2^{p-1}-1},\ C_2 = B^{p-2}\left(1+2^{p-1}\right) > 0 \\ &a = p \geq 2,\ b = 1.\end{aligned}$$

(C) The key lemmas. Our comparisons at infinity are obtained from (some variations of) the next result and its companion Lemma 63, via appropriate choices of the functions $\alpha(t), \beta(t), \rho(t), \lambda(t)$ below.

Lemma 60 *Let $\Omega \subseteq M$ be an unbounded domain, let $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ be such that, for some $D > 0$, the set*

$$\Omega_D = \{x \in \Omega : u(x) - v(x) > D\} \tag{125}$$

is non-empty, with boundary $\partial\Omega_D \subset \Omega$ and $u - v$ is non constant on Ω_D. Suppose that

$$div\left(|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u\right) \geq div\left(|\nabla v|^{-1}\varphi(|\nabla v|)\nabla v\right) \quad on\ \Omega \tag{126}$$

and

$$|\nabla u|, |\nabla v| \leq B \text{ on } \Omega_D.$$

Let $\alpha \in \mathcal{C}^1([D,+\infty)), \beta \in \mathcal{C}^0([D,+\infty))$ *be such that*

$$\alpha(u-v) \geq 0 \tag{127}$$

$$\alpha'(u-v) \geq \beta(u-v) > 0 \tag{128}$$

on Ω_D. *Let also* $\lambda \in \mathcal{C}^1(\mathbb{R})$ *satisfy*

$$\begin{array}{ll} i)\ \lambda(t) \geq 0,\ \forall t \in \mathbb{R}; & ii)\ \lambda(t) = 0,\ \forall t \leq D; \\ iii)\ \lambda'(t) \geq 0,\ \forall t \geq D; & iv)\ \sup_{[D,+\infty)} \lambda \leq 1. \end{array} \tag{129}$$

Then, there exists $R_1 > 0$ *such that, for each* $r > R \geq R_1$

(130)

$$\frac{1}{\int_{B_R \cap \Omega_D} \beta(u-v)\lambda(u-v)\Psi(\nabla u, \nabla v)^a} \geq C\left\{\int_R^r \left[\int_{\partial B_s \cap \Omega_D} \frac{\alpha(u-v)^{\frac{a}{a-b}}}{\beta(u-v)^{\frac{b}{a-b}}}\right]^{-\frac{a-b}{b}} ds\right\}^{\frac{b}{a-b}}$$

for some constant $C > 0$ *independent of* R.

Remark 61 In the above assumptions, Ω_D is necessarily unbounded for otherwise, by the classical comparison principle, we would have $u \leq v + D$ on Ω_D. This observation will not be repeated in the sequel.

Proof. Define the vector field

$$Z = \lambda(u-v)\alpha(u-v)\left\{|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u - |\nabla v|^{-1}\varphi(|\nabla v|)\nabla v\right\}$$

and, using (127), (128), (129) and structural conditions (118), compute

$$divZ \geq C_1\lambda(u-v)\beta(u-v)\Psi(\nabla u, \nabla v)^a.$$

Whence, using the divergence theorem together with (118) and (129), we get

(131)

$$\begin{aligned} & C_1 \int_{\Omega_D \cap B_s} \lambda(u-v)\beta(u-v)\Psi(\nabla u, \nabla v)^a \\ & \leq \int_{\partial(\Omega_D \cap B_s)} < Z, \frac{\partial}{\partial s} > \\ & \leq \int_{\Omega_D \cap \partial B_s} \lambda(u-v)\alpha(u-v)\left||\nabla u|^{-1}\varphi(|\nabla u|)\nabla u - |\nabla v|^{-1}\varphi(|\nabla v|)\nabla v\right| \\ & \leq C_2 \int_{\Omega_D \cap \partial B_s} \lambda(u-v)\alpha(u-v)\Psi(\nabla u, \nabla v)^b. \end{aligned}$$

Now, we apply Holder inequality to the RHS of the above and obtain

(132)

$$\begin{aligned}
&\textstyle\int_{\Omega_D\cap\partial B_s}\lambda(u-v)\alpha(u-v)\Psi\left(\nabla u,\nabla v\right)^b\\
&\qquad\leq\left\{\textstyle\int_{\Omega_D\cap\partial B_s}\lambda(u-v)\beta(u-v)\Psi\left(\nabla u,\nabla v\right)^a\right\}^{\frac{b}{a}}\times\\
&\qquad\times\left\{\int_{\Omega_D\cap\partial B_s}\frac{\alpha(u-v)^{\frac{a}{a-b}}}{\beta(u-v)^{\frac{b}{a-b}}}\right\}^{\frac{a-b}{a}}.
\end{aligned}$$

We set

$$H(s)=\int_{\Omega_D\cap B_s}\lambda(u-v)\beta(u-v)\Psi\left(\nabla u,\nabla v\right)^a$$

$$\xi(s)=\left\{\int_{\Omega_D\cap\partial B_s}\frac{\alpha(u-v)^{\frac{a}{a-b}}}{\beta(u-v)^{\frac{b}{a-b}}}\right\}^{\frac{a-b}{a}}$$

and we note that, by assumptions, $H(s)>0$ for $s>R_0\gg 1$. Moreover, according to the co-area formula,

$$H'(s)=\int_{\Omega_D\cap\partial B_s}\lambda(u-v)\beta(u-v)\Psi\left(\nabla u,\nabla v\right)^a.$$

As a consequence, from (131) and (132) we deduce

$$\frac{H'(s)}{H(s)^{\frac{a}{b}}}\geq\frac{c}{\xi(s)^{\frac{a}{b}}},$$

for some constant $c>0$. Integrating over $[R,r)$ yields the validity of (130). ■

Remark 62 Under the assumption that α,β satisfy (127), (128) on M, the proof of Lemma 60 goes through with M in place of Ω_D and with $\lambda\equiv 1$. The conclusion (130) must be modified accordingly.

In (130) of Lemma 60 we obtain an upper estimate for the quantity

$$\int_{B_R\cap\Omega_D}\beta(u-v)\lambda(u-v)\Psi(\nabla u,\nabla v)^a.$$

Reasoning in a similar way we deduce the following companion lower bound

Lemma 63 *Let Ω,u,v,Ω_D be as in Lemma 60 and let $\rho\in\mathcal{C}^0\left([D,+\infty)\right)$ be a positive, non-constant function such that*

(133) $$\sup_{\Omega_D}\frac{u-v}{\rho(u-v)}=L<+\infty.$$

Let also $\lambda \in \mathcal{C}^1(\mathbb{R})$ (lip. is enough) satisfy

$$\text{(134)} \qquad i)\ \lambda(t) = 0 \text{ if } t \leq D; \quad ii)\ t\lambda'(t) + \lambda(t) > 0 \text{ for } t \geq D.$$

Then, there exists a sufficiently large $R_2 > 0$ such that, for each $r > R \geq R_2$, it holds

(135) .

$$\int_{(B_r \setminus B_R)\cap\Omega_D} \lambda(u-v)\rho(u-v)\,\Psi(\nabla u, \nabla v)^a \geq C \int_R^r \frac{ds}{\left[\int_{\partial B_s\cap\Omega_D} \lambda(u-v)\rho(u-v)\right]^{\frac{a-b}{b}}}$$

The same conclusion holds for constant ρ provided (134) ii) is replaced by $\lambda'(t) > 0$ for every $t > D$.

Proof. Define the vector field

$$Z = \lambda(u-v)(u-v)\left\{|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u - |\nabla v|^{-1}\varphi(|\nabla v|)\nabla v\right\}$$

and compute, using (126) and structural conditions (118)

$$divZ \geq \left(\lambda'(u-v)(u-v) + \lambda(u-v)\right)\Psi(\nabla u, \nabla v)^a.$$

Whence, applying the divergence theorem we get

$$\text{(136)} \qquad \int_{\partial B_s\cap\Omega_D} |Z| \geq \int_{B_s\cap\Omega_D} \left(\lambda'(u-v)(u-v) + \lambda(u-v)\right)\Psi(\nabla u, \nabla v)^a.$$

Note that by (134) and the assumptions on $u-v$, there exists $R_2 > 0$ such that, whenever $s \geq R_2$ it holds

$$RHS\,(136) \geq E$$

for some constant $E > 0$. Therefore, using (133) and the structural conditions once more, from (136) we deduce

$$\frac{E}{L} \leq \int_{\partial B_s\cap\Omega_D} \lambda(u-v)\rho(u-v)\,\Psi(\nabla u, \nabla v)^b.$$

Now, we apply Holder inequality to obtain

$$\frac{E}{L} \leq \left\{\int_{\partial B_s\cap\Omega_D} \lambda(u-v)\rho(u-v)\right\}^{\frac{a-b}{a}} \times \left\{\int_{\partial B_s\cap\Omega_D} \lambda(u-v)\rho(u-v)\,\Psi(\nabla u, \nabla v)^a\right\}^{\frac{b}{a}}.$$

It follows that

$$\int_{\partial B_s\cap\Omega_D} \lambda(u-v)\rho(u-v)\,\Psi(\nabla u, \nabla v)^a \geq \frac{(E/L)^{\frac{a}{b}}}{\left\{\int_{\partial B_s\cap\Omega_D} \lambda(u-v)\rho(u-v)\right\}^{\frac{a-b}{b}}}$$

which in turn integrated over $[R, r] \subset [R_2, +\infty)$ gives (135).

The case where ρ is constant can be dealt with similarly using the vector field

$$Z = \lambda(u - v)\left\{|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u - |\nabla v|^{-1}\varphi(|\nabla v|)\nabla v\right\}.$$

■

(D) The results. To begin with, we establish an L^q-comparison at infinity with $q < +\infty$. Precisely, we give the following

Theorem 64 *Let $\Omega \subset M$ be an unbounded domain and let $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ satisfy*

$$(137) \qquad \begin{cases} div\left(|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u\right) \geq div\left(|\nabla v|^{-1}\varphi(|\nabla v|)\nabla v\right) & on\ \Omega \\ u \leq v\ \ on\ \partial\Omega \end{cases}$$

and

$$(138) \qquad |\nabla u|, |\nabla v| \leq B\ \ on\ \Omega.$$

Assume that, for some $q > \frac{b}{a-b}$,

$$(139) \qquad \frac{1}{\left\{\int_{\partial B_s \cap \Omega}|u - v|^q\right\}^{\frac{a-b}{b}}} \notin L^1(+\infty).$$

If $\partial\Omega \neq \emptyset$ then $u \leq v$ on Ω. If $\partial\Omega = \emptyset$, i.e. $\Omega = M$, then either $u - v = const.$ or $u \leq v$.

Proof. We first consider the case where $\partial\Omega \neq \emptyset$. Assume by contradiction that

$$\{x \in \Omega : u(x) - v(x) > 0\} \neq \emptyset.$$

We choose $D > 0$ so small that

$$\Omega_D = \{x \in \Omega : u(x) > v(x) + D\} \neq \emptyset.$$

Since $u \leq v$ on $\partial\Omega$, $u - v$ is nonconstant on (every connected component of) Ω_D and $\partial\Omega_D \subset \Omega$. We apply Lemma 60 with the choices

$$\alpha(t) = t^{q - \frac{b}{a-b}},\ \beta(t) = \left(q - \frac{b}{a-b}\right)t^{q - \frac{b}{a-b} - 1},\ \ \lambda(t) = \begin{cases} 1 - e^{-(t-D)^2}, \text{ if } t \geq D \\ 0 \text{ if } t < D \end{cases}$$

According to (130) we get, for $R > 0$ large enough, and for some constant $A > 0$,

$$+\infty > \frac{1}{\int_{B_R \cap \Omega_D}\lambda(u - v)(u - v)^{q - \frac{b}{a-b}}} \geq A\left\{\int_R^{+\infty}\frac{ds}{\left[\int_{\partial B_s \cap \Omega}(u - v)^q\right]^{\frac{a-b}{b}}}\right\}^{\frac{b}{a-b}}.$$

This contradicts (139).

Next, we consider the case $\Omega = M$. Again, reasoning by contradiction, we assume that $\{x \in M : u(x) - v(x) > 0\} \neq \emptyset$ and $u - v$ is nonconstant on M. We choose $D > 0$ so small that the set Ω_D is nonempty with (possibly empty) boundary $\partial\Omega_D$. Since, even in this case, $u - v$ cannot be constant on the connected components of Ω_D, we reach a contradiction as above. ■

The next theorem can be considered as an L^∞-comparison over boundaryless domain.

Theorem 65 *Let $u, v \in \mathcal{C}^2(M)$ satisfy*

$$div\left(|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u\right) \geq div\left(|\nabla v|^{-1}\varphi(|\nabla v|)\nabla v\right) \text{ on } M.$$

Assume that

$$u \leq v \text{ on } M$$

and

$$|\nabla u|, |\nabla v| \leq B.$$

If

$$\frac{1}{vol(\partial B_r)^{\frac{a-b}{b}}} \notin L^1(+\infty)$$

then $u - v = A$ on M for some constant $A < 0$.

Proof. We note that if $u - v$ is nonconstant on M, then choosing $\alpha(t) = \beta(t) = e^t$ in the modified version of Lemma 60 given in Remark 62 we get

$$\frac{1}{vol(\partial B_r)^{\frac{a-b}{b}}} \in L^1(+\infty).$$

■

We conclude with an L^∞-comparison which generalizes the results of Langevin-Rosenberg, Miklyukov, Collin-Krust and Hwang mentioned in the introduction.

Theorem 66 *Let $\Omega \subseteq M$ be an unbounded domain and let $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ satisfy (137), (138) above. Assume that*

$$\frac{1}{vol(\partial B_r \cap \Omega)^{\frac{a-b}{b}}} \notin L^1(+\infty) \tag{140}$$

and that

$$\liminf_{r \to +\infty} \frac{\sup_{B_r \cap \Omega}(u - v)}{\int_R^r \frac{1}{vol(\partial B_r \cap \Omega)^{\frac{a-b}{b}}}} = 0, \tag{141}$$

for some $R > 0$ sufficiently large. If $\partial\Omega \neq \emptyset$ then $u \leq v$ on Ω. If $\partial\Omega = \emptyset$, i.e., $\Omega = M$ then $u - v = A$ for some constant $A \in \mathbb{R}$.

Proof. We first consider the case $\partial\Omega \neq \emptyset$. Reasoning by contradiction, we assume that the set

$$\Omega_D = \{x \in \Omega : u(x) > v(x) + D\}$$

is nonempty for some small $D > 0$. Note that $\partial\Omega_D \subset \Omega$ and $u-v$ is nonconstant on every connected component of Ω_D. We apply Lemma 60 and Lemma 63 with the choices

$$\alpha(t) = t;$$
$$\beta(t) = \rho(t) = 1;$$
$$\lambda(t) = \begin{cases} 1 - e^{-(t-D)^2}, \text{ if } t \geq D \\ 0 \text{ if } t < D \end{cases}$$

and obtain, for each $r > R \geq \max(R_1, R_2)$

(142)

$$\int_{B_R\cap\Omega} \lambda(u-v)\,\Psi(\nabla u, \nabla v)^a \leq c_1 \left\{ \int_R^r \left[\int_{\partial B_s \cap \Omega_D} (u-v)^{\frac{a}{a-b}} \right]^{-\frac{a-b}{b}} \right\}^{-\frac{b}{a-b}}$$

(143) $$\int_{(B_r\setminus B_R)\cap\Omega} \lambda(u-v)\,\Psi(\nabla u, \nabla v)^a \geq c_2 \int_R^r \frac{ds}{vol(\partial B_s \cap \Omega)^{\frac{a-b}{b}}}$$

for some appropriate constants $c_1, c_2 > 0$. If we set for brevity

$$I_t = \int_{B_t\cap\Omega} \lambda(u-v)\,\Psi(\nabla u, \nabla v)^a;$$

$$V_R(t) = \int_R^t \frac{ds}{vol(\partial B_s \cap \Omega)^{\frac{a-b}{b}}}$$

$$M(t) = \sup_{B_t\cap\Omega} (u-v).$$

then, from (142) and (143) we deduce, for each $r > R \geq R' = \max(R_1, R_2)$,

$$I_R \leq c_1 \left[\frac{M(r)}{V_R(r)} \right]^{\frac{b}{a-b}} M(r)$$

$$I_r - I_R \geq c_2 V_R(r).$$

The desired contradiction can now be reached arguing as in [Hw1]. Indeed, note that, for each $R''' > R'' > R'$ we have

(144) $$1 \geq \frac{I_{R''} - I_{R'}}{I_{R''}} \geq \frac{c_2}{c_1} \frac{V_{R'}(R'')\,[V_{R''}(R''')]^{\frac{b}{a-b}}}{[M(R''')]^{\frac{b}{a-b}+1}}.$$

Moreover, for each $R''' > R'$ there exists $R'' = R''(R''') \in (R', R''')$ satisfying

(145) $$\text{i) } V_{R'}(R''') = 2V_{R''}(R''') \quad \text{and} \quad \text{ii) } 2V_{R'}(R'') = V_{R'}(R''').$$

Indeed, fix R''' and consider the continuous increasing function

$$t \mapsto F(t) = 2V_{R'}(t) - V_{R'}(R''')$$

defined on $[R', R''']$. Since $F(R') = -V_{R'}(R''') < 0$ and $F(R''') = V_{R'}(R''') > 0$ we conclude the validity of (145) ii). As for condition i) simply use the fact that

$$V_{R'}(R''') = V_{R'}(R'') + V_{R''}(R''').$$

Using (145) into (144) gives, for each $R''' > R'$,

$$1 \geq \frac{c_2}{c_1} 2^{-\frac{b}{a-b}-1} \left[\frac{V_{R'}(R''')}{M(R''')} \right]^{\frac{b}{a-b}+1}$$

showing that

$$\frac{M(r)}{V_{R'}(r)} \geq 2^{-1} \left(\frac{c_2}{c_1} \right)^{\frac{a-b}{a}} > 0, \quad r >> 1.$$

This contradicts (141).

Suppose now that $\Omega = M$. If $u - v$ was nonconstant we could argue as in Theorem 64 and conclude that $u \leq v$ on M. As a consequence, an application of Theorem 65 would give $u - v \equiv A$ for some constant $A < 0$, contradicting our assumption. ■

2.3 A remark on the growth of minimal graphs

(A) Introduction. The previous section began with a brief history on geometric comparison principles for minimal surfaces in $\mathbb{R}^3$. This was the starting point for our comparisons at infinity on Riemannian manifolds involving a more general class of non-linear operators. We would like to conclude the chapter coming back to minimal graphs and spending some words on their asymptotic growth.

As we mentioned above, a result of Bers, [Be] (see also [Sc], [LaRo]) predicts the asymptotic expansion at infinity of a minimal graph over an exterior domain of $\mathbb{R}^2$. In particular, if u satisfies the minimal surface equation on $\{x \in \mathbb{R}^2 : |x| > R > 0\}$ and the graph of u has vertical limiting normal, then the limit

$$a = \lim_{|x| \to +\infty} \frac{u(x)}{\log |x|} \tag{146}$$

exists and is finite. The limiting value $a \in \mathbb{R}$ is named "the logarithmic growth rate of u". To obtain their results Bers and others parametrize the surface over the punctured disc via Enneper-Weierstrass data.

Following [PiRiSe], in Section 2.2 we gave L^p-type estimates for minimal graphs defined on complete manifolds. The L^∞-case was obtained by interpreting the classical term $\log |x|$ as the geometric quantity $\int^{r(x)} \frac{1}{Area(S_t)} dt$, where S_t denotes the circle $\{x \in \mathbb{R}^2 : |x| = t\}$. A version of our L^∞-result sounds as follows. Suppose $(M, \langle , \rangle)$ is a complete m-dimensional manifold. Denote with $B_r, \partial B_r$ the geodesic ball and sphere centered at some fixed point $o \in M$ and of radius $r > 0$. Assume that

$$\frac{1}{Area\,(\partial B_r)} \notin L^1\,(+\infty)\,. \tag{147}$$

If $u \geq c$ defines a non-constant graph outside B_R with non-negative mean curvature and $u|_{\partial B_R} \equiv c$ then (note that $\max_{B_r \setminus \bar{B}_R} u = \max_{\partial B_r} u$)

$$\liminf_{r \to +\infty} \frac{\max_{\partial B_r} u}{\int^r \frac{dt}{Area(\partial B_t)}} > 0. \tag{148}$$

We complete the above picture by means of the following upper estimate

Theorem 67 *Let $(M, \langle , \rangle)$ be an m-dimensional manifold with a pole $o \in M$. Set $r(x) = dist\,(x, o)$ and denote by $B_r, \partial B_r$ the geodesic ball and sphere centered at o and of radius $r > 0$. Assume that the radial sectional curvature of M satisfies*

$$^M K_{rad} \leq 0 \tag{149}$$

Let $u \in \mathcal{C}^2\left(M \setminus \bar{B}_R\right) \cap \mathcal{C}^0\left(M \setminus B_R\right)$, $R > 0$, be such that

$$div\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) \leq 0. \tag{150}$$

Then

$$\limsup_{r\to+\infty}\frac{\min_{\partial B_r} u}{\log r}\leq R. \tag{151}$$

We point out the following immediate consequence in the context of radial graphs over Riemann surfaces. Let us consider a Cartan-Hadamard Riemann surface $(M,\langle,\rangle)$ quasi-isometric to $\mathbb{R}^2$. For instance, we can think of $M=\mathbb{R}^2$ equipped with a (conformal) metric whose Gaussian curvature $K(x)$ satisfies

$$-k(r(x))\leq K(x)\leq 0$$

where $k(t)$ is a non-negative continuous function such that

$$sk\left(s\right)\in L^1\left(+\infty\right).$$

We assume that $c\leq u\in\mathcal{C}^2\left((R,+\infty)\right)\cap\mathcal{C}^0\left([R,+\infty)\right)$ defines a non-constant minimal graph with $u|_{\partial B_R}\equiv c$. Then

$$\bar{R}\leq\frac{u\left(r(x)\right)}{\log\left(r(x)\right)}\leq R,\quad r(x)>>1 \tag{152}$$

for some $\bar{R}>0$. Indeed, since M is quasi-isometric to $\mathbb{R}^2$, we have

$$\frac{r}{volB_r}\geq C\frac{\mathbf{r}}{vol\ \mathbf{B}_r}$$

where $C>0$ is a constant and the "bold" quantities refer to $\mathbb{R}^2$. Therefore

$$\frac{r}{volB_r}\notin L^1\left(+\infty\right)$$

which implies the validity of (147). From (148) we deduce

$$\liminf_{r\to+\infty}\frac{u(r)}{\log r}>0.$$

The upper estimate follows from Theorem 67.

In order to obtain Theorem 67 our viewpoint is to identify the classical term $\log|x|$ with the growth rate of a special surface, the Catenoid, and to consider the desired estimate as the result of a comparison with this surface. Towards this end, in Theorem 69 below, we shall introduce a comparison principle with catenoid-type graphs that generalizes a classic result in minimal surface theory due to Osserman, see Lemma 10.2 in [Os].

(B) Notations. In all that follows we let $(M,\langle,\rangle)$ be a Riemannian manifold of dimension $m\geq 2$ with a pole $o\in M$. We set $r(x)=dist_{(M,\langle,\rangle)}\left(x,o\right)$, so that $r(x)^2\in\mathcal{C}^\infty\left(M\right)$, and we denote by $B_r,\partial B_r$ the geodesic ball and sphere centered at o and of radius $r>0$.

We use the symbols Ω, I to denote any domain of M and any interval in $\mathbb{R}_+$, respectively. Given $u \in \mathcal{C}^2(\Omega)$ the graph $\Gamma_u : \Omega \to M \times \mathbb{R}$ of u over Ω is the imbedding defined by $\Gamma_u(x) = (x, u(x))$. If we equip $M \times \mathbb{R}$ with the product metric $(,) = \langle,\rangle + dt^2$ and Ω with the pull-back metric $\Gamma_u^*(,)$ then the mean curvature H_u (with respect to a chosen normal) of the isometric imbedding Γ_u is expressed by the formula

$$mH_u = div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \tag{153}$$

where $div, \nabla, |.|$ refer to the original metric $\langle,\rangle$ of M.

Given a smooth, m-dimensional model

$$(\mathbf{M}, (,)) = \left([0,+\infty) \times S^{m-1}, d\mathbf{r}^2 + g(\mathbf{r})^2 d\theta^2\right) \tag{154}$$

and $v \in \mathcal{C}^2(I)$, the symbol $\mathbf{H}_v$ stems for the mean curvature of the rotationally symmetric graph of v over $I \times S^{m-1} \subset \mathbf{M}$. Direct computations show that

$$m\mathbf{H}_v = \frac{v''}{(1+(v')^2)^{3/2}} + \frac{v'}{(1+(v')^2)^{1/2}}(m-1)\frac{g'}{g}. \tag{155}$$

Moreover, having fixed $\lambda > 0$, we write v_λ for the function

$$v_\lambda(t) = \lambda v\left(\lambda^{-1}t\right) \tag{156}$$

which is defined on the interval λI. Finally, we agree to devote the special symbol $\mathfrak{C}$ to the function

$$\mathfrak{C}(t) = \cosh^{-1}(g(t)) \geq 0, \quad g(t) \geq 1$$

Thus, in particular, if $g(\mathbf{r}) = \mathbf{r}$ and $\mathbf{M}$ is the standard 2-dimensional Euclidean space, the graph $\mathbf{\Gamma}_{\mathfrak{C}}(\mathbf{r})$ represents the usual upper half-catenoid in $\mathbb{R}^{2+1}$ on the exterior of the unit disk of $\mathbb{R}^2$.

(C) The results. We begin restating in our general setting a comparison principle originally due to Osserman, see [Os].

Lemma 68 *Let $a > 0$ and let $v \in \mathcal{C}^2((a,+\infty)) \cap \mathcal{C}^0([a,+\infty))$ satisfy the following conditions:*

$$i)\ v'(a^+) = +\infty; \quad ii)\ H_{v_\lambda} \geq 0 \text{ on } M \backslash \overline{B_{\lambda a}} \tag{157}$$

for each fixed $\lambda \geq 1$. Let $b > a$. If $u \in \mathcal{C}^2\left(B_b \backslash \overline{B_a}\right) \cap \mathcal{C}^0\left(\overline{B_b} \backslash B_a\right)$ is a solution of the problem

$$\begin{cases} H_u \leq 0 \leq H_v & \text{in } B_b \backslash \overline{B_a} \\ u \geq v & \text{on } \partial B_b \end{cases} \tag{158}$$

then

$$u \geq v \text{ on } \overline{B_b} \backslash B_a. \tag{159}$$

If u is of class $\mathcal{C}^1$ on $B_b \backslash B_a$ (i.e. it has a $\mathcal{C}^1$-extension in a neighborhood of ∂B_a) then the result still holds requiring the validity of (157) ii) only with $\lambda = 1$.

Proof. Fix $\lambda \in (1, \frac{b}{a})$ and set

$$\varepsilon_\lambda = |v_\lambda(b) - v(b)|$$

so that, on ∂B_b,

$$\begin{aligned} v_\lambda - u &= (v_\lambda - v) + (v - u) \\ &\leq \varepsilon_\lambda. \end{aligned}$$

We shall show that

$$v_\lambda - u \leq \varepsilon_\lambda \text{ on } \partial B_{\lambda a}. \tag{160}$$

This latter easy implies the validity of (159). Indeed, by standard comparison we have

$$v_\lambda - u \leq \varepsilon_\lambda \text{ on } \overline{B_b} \backslash B_{\lambda a}$$

and the conclusion follows letting λ tend to 1^+. To prove (160) we reason by contradiction and assume that

$$\varepsilon' = \max_{\partial B_{\lambda a}} (v_\lambda - u) = (v_\lambda - u)(x_0) > \varepsilon_\lambda$$

for some $x_0 \in B_{\lambda a}$. Then $(v_\lambda - u) \leq \varepsilon'$ on $\partial B_b \cup \partial B_{\lambda a}$ and using again a comparison argument we conclude

$$(v_\lambda - u) \leq \varepsilon' \text{ on } \overline{B_b} \backslash B_{\lambda a}.$$

Let γ be the normalized geodesic ray issuing from o and passing through x_0. Since u has a finite gradient on $\partial B_{\lambda a}$ whereas $v'_\lambda(\lambda a^+) = +\infty$, we see that $(v_\lambda - u)$ increases along a small piece of γ inside $B_b \backslash B_{\lambda a}$ near x_0. Therefore, there are points of $B_b \backslash B_{\lambda a}$ where $(v_\lambda - u) > \varepsilon'$, a contradiction.

To conclude, note that dilations of v enable us to overcome the lack of smoothness of u on the inner boundary. In case u has a $\mathcal{C}^1$-extension near ∂B_a the above argument applies directly to $v - u$. ■

Next, we specify the above comparison to catenoid-type graphs.

Theorem 69 *Assume that the radial sectional curvature of M (with respect to its pole o) satisfies*

$${}^M K_{rad} \leq \delta$$

for some constant $\delta \leq 0$. Let s_δ be the solution of the Cauchy problem

$$\begin{cases} s''_\delta + \delta s_\delta = 0 \\ s_\delta(0) = 0;\ s'_\delta(0) = 1. \end{cases}$$

Thus, s_δ defines a smooth m-dimensional model which is the standard Euclidean space if $\delta = 0$ and the Hyperbolic space of constant curvature δ if $\delta < 0$. Set

$$\mathfrak{C}(x) = \cosh^{-1} s_\delta(r(x)) \geq 0, \quad \text{on } M \backslash B_{s_\delta^{-1}(1)}.$$

If $u \in \mathcal{C}^2\left(B_R \backslash \bar{B}_{s_\delta^{-1}(1)}\right) \cap \mathcal{C}^0\left(\bar{B}_R \backslash B_{s_\delta^{-1}(1)}\right) \cap \mathcal{C}^1\left(B_R \backslash B_{s_\delta^{-1}(1)}\right)$, $R > s_\delta^{-1}(1)$, satisfies

$$\begin{cases} \mathit{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \leq 0 & \text{in } B_R \backslash \bar{B}_{s_\delta^{-1}(1)} \\ u \geq \mathfrak{C} & \text{on } \partial B_R. \end{cases}$$

then

$$u \geq \mathfrak{C} \quad \text{in } \bar{B}_R \backslash B_{s_\delta^{-1}(1)}.$$

In case $\delta = 0$, the $\mathcal{C}^1$-regularity condition on ∂B_1 can be omitted; moreover, the result can be extended to every annulus $B_R \backslash B_{R_0}$, $R > R_0 > 0$, up to using $\mathfrak{C}_{R_0}(r)$ instead of $\mathfrak{C}$.

Proof. We shall show that the assumptions of Lemma 68 are fulfilled with $v = \mathfrak{C}$ and $a = s_\delta^{-1}(1)$. Clearly, by definition,

$$\mathfrak{C}'\left(s_\delta^{-1}(1)^+\right) = +\infty.$$

As for the remaining properties, we divide the verification in two steps.
First step. A direct computation shows

$$m\mathbf{H}_\mathfrak{C} = \frac{s_\delta s_\delta''\left(s_\delta^2 - 1\right) + s_\delta'^2\left[(m-1)\left(s_\delta'^2 - 1\right) + (m-2)\, s_\delta^2\right]}{s_\delta\left[(s_\delta^2 - 1) + s_\delta'^2\right]^{3/2}}.$$

Suppose $\delta = 0$. Then, we have

$$\mathbf{H}_\mathfrak{C}(\mathbf{r}) = \frac{(m-2)}{m}\mathbf{r}^{-2} \geq 0.$$

Moreover, having fixed any $\lambda > 0$,

$$\mathbf{H}_{\mathfrak{C}_\lambda}(\mathbf{r}) = \lambda^{-1}\mathbf{H}_\mathfrak{C}\left(\lambda^{-1}\mathbf{r}\right). \tag{161}$$

Suppose now that $\delta < 0$. Using the basic relations $s_\delta'^2 + \delta s_\delta^2 = 1$ and $s_\delta'' + \delta s_\delta = 0$, we obtain

$$\mathbf{H}_\mathfrak{C} = \frac{\left[-\delta s_\delta^2 (m-1)(1-\delta) - \delta(m-1) + (m-2)\right]}{m s_\delta^2 (1-\delta)^{3/2}} \geq 0.$$

Second step. Recall that, for $v(x) = v(r(x))$,

$$mH_v = \frac{v''}{(1+(v')^2)^{3/2}} + \frac{v'}{(1+(v')^2)^{1/2}}\Delta r.$$

Therefore, if $v' \geq 0$, from the Hessian comparison theorem we deduce

$$H_v(r(r)) \geq \mathbf{H}_v(r(x)).$$

When specialized to $v = \mathfrak{C}$ this latter yields

$$H_{\mathfrak{C}} \geq \mathbf{H}_{\mathfrak{C}} \geq 0.$$

Moreover, if $\delta = 0$, we have for any fixed $\lambda > 0$,

$$H_{\mathfrak{C}_\lambda} \geq \mathbf{H}_{\mathfrak{C}_\lambda} \geq 0.$$

■

Remark 70 We point out the role of (161): on one hand, as explained above, by means of this condition we can overcome the lack of smoothness of u near the inner boundary; on the other hand, up to using $v = \mathfrak{C}_a$ instead of $\mathfrak{C}$, we can consider graphs defined outside the ball B_a, for any $a > 0$.

We are now in position to prove Theorem 67.

Proof (of Theorem 67). We shall use Euclidean type catenoid $\mathfrak{C}(x) = \cosh^{-1} r(x)$. Without loss of generality, we assume $R = 1$. By contradiction, suppose that, along a sequence $\{r_n\} \nearrow +\infty$,

$$\frac{\min_{\partial B_{r_n}} u}{\log(r_n)} \geq (1+\delta)$$

for some $\delta > 0$. Fix $c \geq 0$ in such a way that

$$u - c < \mathfrak{C} \quad \text{on } \partial B_1. \tag{162}$$

Note that, for $n >> 1$,

$$\min_{\partial B_{r_n}} (u - c) \geq \left(1 + \frac{\delta}{2}\right) \mathfrak{C}(r_n).$$

Since

$$\begin{cases} H_{u-c} \leq 0 \leq H_{\mathfrak{C}} & \text{on } B_{r_n} \backslash \bar{B}_1 \\ u - c > \mathfrak{C} & \text{on } \partial B_{r_n} \end{cases}$$

an application of Theorem 67 with $\delta = 0$ gives

$$u - c \geq \mathfrak{C} \quad \text{on } \partial B_1$$

contradicting (162). ■

Remark 71 We would like to use a similar technique to obtain information on the asymptotic behavior of exterior minimal graphs when ${}^M K_{rad} \leq \delta < 0$. The behavior should be sensitive of the bound δ. Comparing with $\mathfrak{C}$ -graphs does not work. It is possible that a different reference graph behave better. The answer should come from a careful study of the hyperbolic space.

Remark 72 When $m \geq 3$, also the case $\delta = 0$ should be improved by a more appropriate choice of the reference graph.

Remark 73 A systematic extension of the technique to more general non-linear operators would be welcome.

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