POTENTIAL THEORY FOR MANIFOLDS WITH BOUNDARY AND APPLICATIONS TO CONTROLLED MEAN CURVATURE GRAPHS

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Abstract. In this paper we characterize the Neumann-parabolicity of manifolds with boundary in terms of a new form of the classical Ahlfors maximum principle and of a version of the so called Kelvin-Nevanlinna-Royden criterion.

The motivation underlying this study is to obtain new information on the geometry of graphs with prescribed mean curvature inside a Riemannian product of the type $N \times \mathbb{R}$. In this direction two kind of results will be presented: height estimates for constant mean curvature graphs parametrized over unbounded domains in a complete manifold, which extend results by A. Ros and H. Rosenberg valid for domains of $\mathbb{R}^2$, and slice type results for graphs whose superlevel sets have finite volume. Finally, the use of the Ahlfors maximum principle allows us to establish a connection between the Neumann parabolicity and the Dirichlet parabolicity commonly used in minimal surface theory. In particular we will be able to give a deterministic proof of special cases of a result by R. Neel

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Introduction

The aim of this paper is twofold: on the one hand, it gives a contribution to the potential theory of Riemannian manifolds with boundary, presenting new aspects of the theory and emphasizing new connections with other potential theoretic notions in the literature. On the other hand, it contains new results on graphs of prescribed mean curvature, parametrized over a non-compact domain. The potential theoretic viewpoint will enable us to get these geometric conclusions in a quite direct way, showing how much the techniques are effective.

The first main goal is to extend to non-compact Riemannian manifolds with boundary the use of two important tools in the geometric analysis of compact spaces, namely, the integration by parts and the weak maximum principle for subharmonic functions. We shall see that these are intimately related, and in fact equivalent, to the parabolicity of the manifold with respect to Neumann boundary conditions. Said differently, the reflected Brownian motion is recurrent. In some sense, this definition of parabolicity follows the most traditional paths, [9, 10, 11, 12]. Another natural notion of parabolicity is obtained by imposing Dirichlet boundary conditions, namely, by requiring that, almost surely, the Brownian motion hits the boundary of the manifolds in finite time, and dies. This notion of parabolicity turns out to be very fruitful in the investigation of minimal surfaces in $\mathbb{R}^3$ or, more generally, in product three manifolds $N \times \mathbb{R}$, [6, 26, 22, 24]. Our maximum principle viewpoint of parabolicity will enable us to prove that these two definitions are in a certain hierarchy. More precisely, the Neumann condition gives rise to the strongest of the notions of parabolic manifolds. On the other hand it is also the one which seems to be more related to the geometry of the underlying space because, for instance, it has a direct link with volume growth properties. As an application of this order of ideas, we are able to deduce that every proper minimal graph with zero boundary values over a smooth domain of $\mathbb{R}^2$ is parabolic in our traditional sense, hence in the Dirichlet sense, because of its area growth property.

The second main goal of the paper, which in fact has primarily motivated our study of potential theoretic properties of manifolds with boundary, is the attempt to obtain new information on the geometry of graphs with boundary and prescribed mean curvature inside a Riemannian product of the type $N \times \mathbb{R}$. Indeed, parabolicity is a kind of compactness from many viewpoints; see e.g. the account in [28] for the case of manifolds without boundary. The maximum principle viewpoint enables us to produce, in a very straightforward way, sharp a-priori height estimates for positive CMC graphs over unbounded and non-homogeneous domain manifolds. This investigation fits into the long tradition initiated in the seventies with the pioneering works by J. Serrin, [33], for compact domains of $\mathbb{R}^2$ and culminated in the very recent work by A. Ros and H. Rosenberg, [31], for unbounded domains. The original proof of this latter result makes an essential use of the homogeneity of the Euclidean space in combination with the convergence theory of CMC surfaces. Apparently, such a method cannot be extended directly to general manifolds. On the other hand, the Stokes theorem viewpoint of parabolicity leads naturally to the detection of new interesting phenoma for graphs of non-positive curvature, which are related to the volume of their superlevel sets.

In order to put the precise definition of parabolicity we need to recall the notion of weak sub (super) solution subjected to Neumann boundary conditions. This, in turn, relies on a suitable notion of local Sobolev functions on a manifold with boundary.

Let $(M, \langle\, ,\rangle)$ be an oriented Riemannian manifold with smooth boundary $\partial M \neq \emptyset$ and exterior unit normal ν. The interior of M (as a manifold with boundary) is denoted by $\mathrm{int}M = M \setminus \partial M$. By a domain in M we mean a non-necessarily connected open set $D \subseteq M$. We say that the domain D is smooth if its topological boundary ∂D is a smooth hypersurface Γ with boundary $\partial\Gamma = \partial D \cap \partial M$. Clearly, if $\partial M = \emptyset$ then the smoothness condition reduces to the usual one. It is a standard fact that every manifold M with (possibly empty) boundary has an exhaustion by smooth pre-compact domains. Simply choose a proper smooth function $\rho : M \to \mathbb{R}_{\geq 0}$ (which in turn can be defined by $\rho = \sum_j j\rho_j$, where $\{\rho_j\}$ is a countable smooth partition of unit with compact supports) and, according to Sard theorem, take a sequence $\{t_k\} \nearrow +\infty$ such that t_k is a regular value for both $\rho|_{\mathrm{int}M}$ and $\rho|_{\partial M}$. Then $D_k = \{\rho < t_k\}$ defines the desired exhaustion with smooth boundary $\partial D_k = \{\rho = t_k\}$. For completeness we recall that a proper smooth fun Adopting a notation similar to the one in [10], for any domain $D \subseteq M$ we define

$$\partial_0 D = \partial D \cap \mathrm{int}M.$$

This can be called the *Dirichlet boundary* of the domain. Note also that D could include part of the boundary of M. We therefore set

$$\partial_1 D = \partial M \cap D,$$

that can be called the *Neumann boundary* of D (although it is not in the topological boundary of the domain). Finally, the *interior part* of D, in the sense of manifolds with boundary, is defined as

$$\mathrm{int}D = D \cap \mathrm{int}M,$$

so that, in particular,

$$D = \mathrm{int}D \cup \partial_1 D.$$

We recall that the Sobolev space $W^{1,2}(\mathrm{int}D)$ is defined as the Banach space of functions $u \in L^2(\mathrm{int}D)$ whose distributional gradient satisfies $\nabla u \in L^2(\mathrm{int}D)$. By the Meyers-Serrin density result, [19, Theorem 10.15], this space coincides with the closure of $C^\infty(\mathrm{int}D)$ with respect to the usual

Sobolev norm $\|u\|_{W^{1,2}} = \|u\|_{L^2} + \|\nabla u\|_{L^2}$. In fact, it can be shown that $W^{1,2}(\mathrm{int}D)$ coincides with the $W^{1,2}$-closure of the space $C^\infty(D) \cap W^{1,2}(\mathrm{int}D)$.

Moreover, if $u \in W^{1,2}(\mathrm{int}D)$ is compactly supported in D then it can be approximated by functions that are smooth up to $\partial_1 D$ and with compact support in an arbitrarily small neighborhood of $\mathrm{supp}u$. It follows that if $D = M$ and M is complete $W^{1,2}(\mathrm{int}M)$ can be also realised as the $W^{1,2}$-closure of $C_c^\infty(M)$; see Appendix A.

We will denote with $W_0^{1,2}(D)$ the the $W^{1,2}$-closure of $C_c^\infty(D)$. Note that if $D = M$ is complete, then $W_0^{1,2}(M) = W^{1,2}(\mathrm{int}M)$. On the other hand, if $D \neq M$, $W_0^{1,2}(D)$ is the space of all the Sobolev functions $u \in W^{1,2}(\mathrm{int}(D))$ satisfying the Dirichlet boundary condition $u = 0$ on $\partial_0 D$.

The space $W^{1,2}_{\mathrm{loc}}(\mathrm{int}D)$ of local $W^{1,2}$-Sobolev functions is classically defined by the condition that $u \cdot \chi \in W^{1,2}(\mathrm{int}(D))$ for every cut-off function $\chi \in C_c^\infty(\mathrm{int}D)$. This turns out to be equivalent to the fact that $u \in W^{1,2}(\Omega)$, for every $\Omega \subset\subset \mathrm{int}D$. We extend this notion by including the Neumann boundary of the domain: this is vital in order to introduce a distributional meaning of (sub-)solutions of the Neumann problem. Accordingly, and with a slight abuse of notation, we let

$$W^{1,2}_{\mathrm{loc}}(D) = \{u \in W^{1,2}(\mathrm{int}\Omega),\ \forall \text{domain } \Omega \subset\subset D = \mathrm{int}D \cup \partial_1 D\}.$$

Now, suppose $D \subseteq M$ is any domain. We put the following

Definition 0.1. *By a weak Neumann sub-solution $u \in W^{1,2}_{loc}(D)$ of the Laplace equation, i.e., a weak solution of the problem*

$$\begin{cases} \Delta u \geq 0 & \text{on } \mathrm{int}D \\ \dfrac{\partial u}{\partial \nu} \leq 0 & \text{on } \partial_1 D, \end{cases} \tag{1}$$

we mean that the following inequality

$$-\int_{\mathrm{int}D} \langle \nabla u, \nabla \varphi \rangle \geq 0 \tag{2}$$

holds for every $0 \leq \varphi \in C_c^\infty(D)$. Actually, since $\partial_1 D$ has measure zero, we can always think of u and ∇u as (a.e.) defined on the whole domain D and write

$$-\int_D \langle \nabla u, \nabla \varphi \rangle \geq 0.$$

Similarly, by taking $D = M$, one defines the notion of weak Neumann sub-solution of the Laplace equation on M as a function $u \in W^{1,2}_{loc}(M)$ which satisfies (2) for every $0 \leq \varphi \in C_c^\infty(M)$. As usual, the notion of weak supersolution can be obtained by reversing the inequality and, finally, we speak of a weak solution when the equality holds in (2) without any sign condition on φ.

Remark 0.2. Clearly, in the above definition, it is equivalent to require that (2) holds for every $0 \leq \varphi \in \mathrm{Lip}_c(D)$. Moreover, are recalled above, standard density arguments work even for manifolds with boundary and, therefore, (2) extends to all compactly supported $0 \leq \varphi \in W_0^{1,2}(D)$.

Remark 0.3. Note that in the equality case we have the usual notion of variational solution of the mixed problem

$$\begin{cases} \Delta u = 0 & \text{on } \mathrm{int} D \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial_1 D \\ u = 0 & \text{on } \partial_0 D. \end{cases}$$

Remark 0.4. If $\partial M = \emptyset$ or, more generally, $D \subseteq \mathrm{int} M$, the Neumann condition disappears and we recover the usual definition of weak sub-(super-)solution. Obviously, in the smooth setting, a classical solution of (1) is also a weak Neumann subsolution as one can verify using integration by parts. Actually, this is true in a more general setting. See Definition 4.3 and Lemma 4.4 in Subsection 4.1.

We are now ready to give the following definition of parabolicity in the form of a Liouville-type result.

Definition 0.5. *An oriented Riemannian manifold M with boundary $\partial M \neq \emptyset$ is said to be parabolic if any bounded above, weak Neumann subsolution of the Laplace equation on M must be constant. Explicitly, for every $u \in C^0(M) \cap W_{loc}^{1,2}(M)$,*

$$\begin{cases} \Delta u \geq 0 & \text{on } \mathrm{int} M \\ \dfrac{\partial u}{\partial \nu} \leq 0 & \text{on } \partial M \\ \sup_M u < +\infty \end{cases} \Rightarrow \quad u \equiv \text{const.} \tag{3}$$

Remark 0.6. In case $\partial M = \emptyset$ the Neumann condition in (3) is void and, in this sense, it can be considered as trivially satisfied. Moreover, the orientability assumption, which is needed to guarantee the existence of the exterior unit normal ν, becomes unnecessary and we recover the usual definition of a parabolic manifold without boundary in terms of the Liouville property for sub-harmonic functions.

It is known from [10] that, if M is complete with respect to the intrinsic distance function d, then geometric conditions implying parabolicity rely on volume growth properties of the space. In order to give the precise statement it is convenient to introduce some notation. Having fixed a reference origin $o \in \mathrm{int} M$, we set $B_R^M(o) = \{x \in M : d(x, o) < R\}$ and $\partial B_R^M(o) = \{x \in M : d(x, o) = R\}$, the metric ball and sphere of M centered at o and of radius $R > 0$. We also denote by $r(x) = d(x, o)$ the distance function from o. Clearly, $r(x)$ is Lipschitz, hence differentiable a.e. in $\mathrm{int} M$. Moreover, for a.e. $x \in \mathrm{int} M$, differentiating r along a minimizing geodesic from o to x (which exists by completeness) we easily see that the usual Gauss

Lemma holds, namely, $|\nabla r| = 1$ a.e. in $\mathrm{int} M$. Therefore, by the co-area formula applied to $r|_{\mathrm{int}M}$ and the fact that $\mathrm{vol} B_R^M(o) = \mathrm{vol}\left(B_R^M(o) \cap \mathrm{int} M\right)$, we have

$$\frac{d}{dR}\mathrm{vol} B_R^M(o) = \mathrm{Area}\left(\partial_0 B_R^M(o)\right),$$

for a.e. $R > 0$.

The following result is due to Grigor'yan [10]. For a PDEs proof in the C^1 case see Theorem 4.6 and Remark 4.8.

Theorem 0.7. *Let $(M, \langle\, ,\rangle)$ be a complete Riemannian manifold with boundary $\partial M \neq \emptyset$. If, for some reference point $o \in M$,*

$$\int^{+\infty} \frac{1}{\mathrm{Area}\left(\partial_0 B_R^M(o)\right)} = +\infty$$

then M is parabolic.

Remark 0.8. It is a usual consequence of the co-area formula that the area growth condition is implied by the volume growth condition

$$\int^{+\infty} \frac{R}{\mathrm{vol} B_R^M(o)} = +\infty.$$

This latter, in turn, follows from a quadratic volume growth assumption

$$\mathrm{vol} B_R^M(o) = \mathcal{O}(R^2), \text{ as } R \to +\infty.$$

We also recall that the volume growth condition is more stable with respect to (even rough) perturbations of the metric and sometimes it characterizes the parabolicity of the space.

The first main result of the paper is the following maximum principle characterization of parabolicity. It extends to manifolds with boundary a classical result by L.V. Ahlfors.

Theorem 0.9 (Ahlfors maximum principle)**.** *M is parabolic if and only the following maximum principle holds. For every domain $D \subseteq M$ with $\partial_0 D \neq \emptyset$ and for every $u \in C^0\left(\overline{D}\right) \cap W^{1,2}_{loc}(D)$ satisfying*

$$\begin{cases} \Delta u \geq 0 & \text{on } \mathrm{int} D \\ \dfrac{\partial u}{\partial \nu} \leq 0 & \text{on } \partial_1 D \\ \sup_D u < +\infty \end{cases}$$

in the weak sense, it holds

$$\sup_D u = \sup_{\partial_0 D} u.$$

It is worth to observe that, when $D = M$, the Neumann boundary condition plays no role and the result takes the following form which is crucial in the applications. This global maximum principle property of subharmonic functions was adopted by F.R. De Lima [7] as a definition of a weak notion of parabolicity for manifolds with boundary; see Section 3.

Theorem 0.10. *Let $(M, \langle\,,\rangle)$ be a parabolic manifold with boundary $\partial M \neq \emptyset$. If $u \in C^0(M) \cap W^{1,2}_{loc}(\mathrm{int}M)$ satisfies*

$$\begin{cases} \Delta u \geq 0 & \text{on } \mathrm{int}M \\ \sup_M u < +\infty \end{cases}$$

then

$$\sup_M u = \sup_{\partial M} u.$$

It is not surprising that this global maximum principle proves to be very useful to get height estimates for constant mean curvature hypersurfaces in product spaces. By way of example, we point out the following

Theorem 0.11 (Height estimate)**.** *Let $(N, \langle\,,\rangle)$ be a Riemannian manifold without boundary and Ricci curvature satisfying $Ric_N \geq 0$. Let Σ be a complete, oriented hypersurface in $N \times \mathbb{R}$ with boundary $\partial\Sigma \neq \emptyset$ and satisfying the following requirements:*

(i) *Σ has quadratic intrinsic volume growth*

$$\mathrm{vol}B^{\Sigma}_R(o) = \mathcal{O}\left(R^2\right), \text{ as } R \to +\infty; \tag{4}$$

(ii) *$\partial\Sigma$ is contained in the slice $N \times \{0\}$;*

(iii) *For a suitable choice of the Gauss map $\mathcal{N}$ of Σ, the hypersurface Σ has constant mean curvature $H > 0$ and the angle Θ between $\mathcal{N}$ and the vertical vector field $\partial/\partial t$ is contained in the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$, i.e.,*

$$\cos\Theta = \left\langle \mathcal{N}, \frac{\partial}{\partial t} \right\rangle \leq 0.$$

If Σ is contained in a slab $N \times [-T, T]$ for some $T > 0$, then

$$\Sigma \subseteq N \times \left[0, \frac{1}{H}\right].$$

We observe explicitly that (4) can be replaced by the stronger extrinsic condition

$$\mathrm{vol}\left(B^N_R(o) \cap \Sigma\right) = \mathcal{O}(R^2), \text{ as } R \to +\infty,$$

which, in turn, follows from the relation

$$B^{\Sigma}_R(o) \subseteq B^N_R(o) \cap \Sigma.$$

We also note that there are important situations where the assumption on the Gauss map is automatically satisfied and the volume growth condition on the hypersurface is inherited from that of the ambient space. The following height estimate extends previous results for H-graphs over non-compact domains, [33, 14, 4, 34, 31]. We recall that a *proper graph* in $N \times \mathbb{R}$ is a (topological) graphical surface with boundary $\Sigma = \{(x, u(x)) : x \in \overline{\Omega}\}$ parametrized by a function $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ over a domain $\Omega \subset N$. The (topological) boundary of Σ is clearly given by $\partial\Sigma = \{(x, u(x)) : x \in \partial\Omega\}$, whereas its interior $\mathrm{int}\Sigma = \{(x, u(x)) : x \in \Omega\}$ is a smooth hypersurface of $N \times \mathbb{R}$. Say that the proper graph Σ is an H-graph (or a graph with constant mean curvature H) if the smooth hypersurface intΣ has constant mean curvature H with respect to a chosen Gauss map.

Theorem 0.12 (Height estimate for graphs)**.** *Let $(N, \langle\, ,\rangle)$ be a complete, Riemannian manifold without boundary satisfying $Ric_N \geq 0$ and*

$$\mathrm{vol} B_R^N(o) = \mathcal{O}(R^2), \ \ \textit{as } R \to +\infty.$$

Let $M \subset N$ be a closed domain with smooth boundary $\partial M \neq \emptyset$. Suppose we are given a proper graph Σ over M with boundary $\partial\Sigma \subset N \times \{0\}$ and constant mean curvature $H > 0$ with respect to the downward Gauss map. If Σ is contained in a vertical slab, then

$$\Sigma \subseteq N \times \left[0, \frac{1}{H}\right].$$

Remark 0.13. As it will be clear from the proof, the volume growth condition of N can be replaced by the more natural assumption $\mathrm{vol}(B_R^N(o) \cap M) = \mathcal{O}(R^2)$. This means that, from the viewpoint of volumes, the domain M is small enough. See Lemma 2.1 and Remark 2.2

In the particular case of graphs over a domain of a surface of non-negative Gauss curvature we obtain the following result that extends, with a different proof, Theorem 4 in [31].

Corollary 0.14. *Let $(N, \langle\, ,\rangle)$ be a complete 2-dimensional Riemannian manifold without boundary of non-negative Gauss curvature. Let $M \subset N$ be a closed domain with smooth boundary $\partial M \neq \emptyset$. Suppose we are given a proper graph Σ over N with boundary $\partial\Sigma \subset N \times \{0\}$ and constant mean curvature $H > 0$ with respect to the downward Gauss map. Then*

$$\Sigma \subseteq N \times \left[0, \frac{1}{H}\right].$$

In the setting of manifolds without boundary, it is well known from a classical work by T. Lyons and D. Sullivan [23] that the validity of an L^2-divergence theorem is related, and in fact equivalent, to the parabolicity of the space. We shall complete the picture by extending the L^2-divergence theorem to non-compact manifolds with boundary.

Theorem 0.15 (L^2-divergence theorem)**.** *Let $(M, \langle\, ,\rangle)$ be a Riemannian manifold with boundary $\partial M \neq \emptyset$ and outward pointing unit normal ν. Then M is parabolic if and only if the following holds. Let X be a vector field on M satisfying the following conditions:*

$$\begin{aligned} &\textit{(a)}\ |X| \in L^2(M) \\ &\textit{(b)}\ \langle X, \nu\rangle \in L^1(\partial M) \\ &\textit{(c)}\ \operatorname{div} X \in L^1_{loc}(M),\ (\operatorname{div} X)_- \in L^1(M). \end{aligned} \tag{5}$$

Then

$$\int_M \operatorname{div} X = \int_{\partial M} \langle X, \nu\rangle.$$

A weaker version of the L^2-divergence theorem, involving solutions X of inequalities of the type $\operatorname{div} X \geq f$ with boundary conditions $\langle X, \nu \rangle \leq 0$, will be employed in our investigations on hypersurfaces in product spaces; see Proposition 4.5. In particular, from this latter we shall obtain the following result for hypersurfaces contained in a half-space of $N \times \mathbb{R}$.

Theorem 0.16 (Slice theorem)**.** *Let $(N, \langle\, , \rangle)$ be a Riemannian manifold without boundary. Let $\Sigma \subset N \times [0, +\infty)$ be a complete, oriented hypersurface with boundary $\partial\Sigma \neq \emptyset$ contained in the slice $N \times \{0\}$ and satisfying the volume growth condition*

$$\operatorname{vol} B_R^{\Sigma}(o) = \mathcal{O}(R^2), \text{ as } R \to +\infty.$$

Assume that, for a suitable choice of the Gauss map $\mathcal{N}$ of Σ, the hypersurface Σ has non-positive mean curvature $H(x) \leq 0$ and the angle Θ between $\mathcal{N}$ and the vertical vector field $\partial/\partial t$ is contained in the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$, i.e.,

$$\cos\Theta = \left\langle \mathcal{N}, \frac{\partial}{\partial t} \right\rangle \leq 0.$$

If there exists some half-space $N \times [t, +\infty)$ of $N \times \mathbb{R}$ such that

$$\operatorname{vol}\left(\Sigma \cap \left(N \times [t, +\infty)\right)\right) < +\infty,$$

then $\Sigma \subset N \times \{0\}$.

When Σ is given graphically over a manifold M with quadratic volume growth, we shall obtain the following variant of the slice theorem that involves the volumes of orthogonal projections of Σ on M. Its proof requires a Liouville-type theorem for the mean curvature operator under volume growth conditions; see Theorem 4.6.

Theorem 0.17 (Slice theorem for graphs)**.** *Let $(M, \langle\, , \rangle)$ be a complete, oriented manifold with boundary $\partial M \neq \emptyset$, outward pointing unit normal ν, and (at most) quadratic volume growth, i.e.,*

$$\operatorname{vol} B_R^{M}(o) = \mathcal{O}(R^2), \text{ as } R \to +\infty,$$

for some origin $o \in M$. Let Σ be a graph over M with non-positive mean curvature $H(x) \leq 0$ with respect to the orientation given by the downward pointing Gauss map $\mathcal{N}(x)$. Assume that $\partial\Sigma \cap (M \times \{T\}) = \emptyset$ for some $T > 0$ and that at least one of the following conditions is satisfied:

- (a) *$\partial\Sigma \subset M \times \{0\}$ and $\Sigma \subset M \times [0, +\infty)$.*
- (b) *M and Σ are real analytic.*
- (c) *On $\partial\Sigma$, the Gauss map $\mathcal{N}(x)$ of Σ and the Gauss map $\mathcal{N}_0(x) = (-\nu(x), 0)$ of the boundary $\partial M \times \{t\}$ of any slice form an angle $\theta(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.*

If the portion of the graph Σ contained in some half-space $M \times [t, +\infty)$ has finite volume projection on the slice $M \times \{0\}$, then Σ is a horizontal slice of $M \times \mathbb{R}$.

It is worth to point out that, in the setting of manifolds without boundary and for $H = 0$, half-space properties in a spirit similar to our slice-type theorems have been obtained in the very recent paper [32] by H. Rosenberg, F. Schulze and J. Spruck. More precisely, they are able to show that curvature restrictions and potential theoretic properties (parabolicity) of the base manifold M in the ambient product space $M \times \mathbb{R}$ force properly immersed minimal hypersurfaces and entire minimal graphs in a half-space to be totally geodesic slices. This holds without any further condition on their superlevel sets.

The paper is organized as follows. In Section 1 we recall the link between parabolicity and absolute capacity of compact subsets. We also take the occasion to give a detailed proof of the existence and regularity of the equilibrium potentials of condensers in the setting of manifolds with boundary. These rely on the solution of mixed boundary value problems in non-smooth domains. Section 2 contains the proof of the maximum principle characterization of parabolicity and its applications to obtain height estimates for complete CMC hypersurfaces with boundary into Riemannian products. In Section 3 we survey, and compare, different notions of parabolicity for manifolds with boundary. In Section 4 we relate the parabolicity of a manifold with boundary to the validity of the L^2-Stokes theorem. We also provide a weak form of this result that applies to get slice-type results for hypersurfaces with boundary in Riemannian products. Further slice-type results that are based on Liouville-type theorem for graphs are also given.

In conclusion of this introductory part we mention that there are natural and interesting applications and extensions of the the results obtained in this paper both to Killing graphs and to the p-Laplace operator.

These aspects will be presented in the forthcoming papers [17] and [18], respectively.

1. Capacity & equilibrium potentials

As in the case where M has no boundary, given a compact set K and an open set Ω containing K the capacity of the condenser (K, Ω) is defined by

$$\operatorname{cap}(K,\Omega) = \inf\left\{ \int_\Omega |\nabla u|^2 \,:\, u \in C^\infty_c(\Omega)\,, u \geq 1 \text{ on } K \right\}.$$

When $\Omega = M$, we write $\operatorname{cap}(K, M) = \operatorname{cap}(K)$ and we refer to it as the (absolute) capacity of K.

A simple approximation argument shows that the infimum on the right hand side can be equivalently computed letting u range over the set

$$\{u \in \mathrm{Lip}_c(\Omega) \,:\, u = 1 \text{ on } K\}$$

or even over

$$W_0(K,\Omega) = \{u \in C(\overline{\Omega}) \cap W^{1,2}_0(\Omega) \,:\, u = 1 \text{ on } K\}.$$

We refer to functions in $W_0(K,\Omega)$ as admissible potentials for the condenser (K,Ω).

The usual monotonicity properties of capacity hold, namely, if $K \subseteq K_1$ are compact sets and $\Omega \subseteq \Omega_1$ are open, then $\mathrm{cap}(K,\Omega_1) \leq \mathrm{cap}(K_1,\Omega_1) \leq \mathrm{cap}(K_1,\Omega)$ and this allows to define first the capacity of an open set $U \subset \Omega$ as $\mathrm{cap}(U,\Omega) = \sup_{U\supset K,\ \mathrm{compact}} \mathrm{cap}(K,\Omega)$ and then the capacity of an arbitrary set $E \subset \Omega$ as $\mathrm{cap}(E,\Omega) = \inf_{E\subset U\mathrm{open}} \mathrm{cap}(U,\Omega)$.

We are going to show that the Liouville-type definition of parabolicity given in the introduction is equivalent to the statement that every compact subset has zero capacity. This depends on the construction of equilibrium potentials for capacity, which plays a vital role also in the proof of the L^2 divergence theorem characterization of parabolicity, Theorem 0.15. It should be pointed out that while these results are in some sense well known, we haven't been able to find a reference which deals explicitly with matters concerning regularity up to the boundary of these equilibrium potentials.

The following simple lemma will be useful in the proof of the proposition.

Lemma 1.1. *Let $D \Subset \Omega$ be open sets, and let D_n and Ω_n be a sequence of open sets such that*

$$\overline{D} \subseteq D_{n+1} \subseteq D_n \subseteq \overline{D}_n \Subset \Omega_n \subseteq \Omega_{n+1} \subseteq \Omega, \quad \cap_n \overline{D}_n = \overline{D}, \quad \cup_n \Omega_n = \Omega.$$

Then

$$\lim_n \mathrm{cap}(\overline{D}_n, \Omega_n) = \mathrm{cap}(\overline{D}, \Omega). \tag{6}$$

Proof. It follows from monotonicity that, for every n, $\mathrm{cap}(\overline{D}_n,\Omega_n)$ is monotonically decreasing and greater than or equal to $\mathrm{cap}(\overline{D},\Omega)$ so the limit on the left hand side of (6) exists and

$$\lim_n \mathrm{cap}(\overline{D}_n, \Omega_n) \geq \mathrm{cap}(\overline{D}, \Omega).$$

For the converse, let $\phi \in \mathrm{Lip}_c(\Omega)$ with $\phi = 1$ on $\overline{D}$, and for $\varepsilon > 0$ let

$$\phi_\varepsilon = \min\left\{1, \left(\frac{\phi - \varepsilon}{1 - 2\varepsilon}\right)_+\right\}.$$

By assumption, for every sufficiently large n we have

$$\overline{D}_n \subseteq \{x : 1 - \varepsilon \leq \phi(x) \leq 1\}$$

and

$$\{x : \varepsilon \leq \phi(x) \leq 1\} \subset \Omega_n,$$

and therefore ϕ_ε is an admissible potential for the condenser $(\overline{D}_n, \Omega_n)$ so that

$$\int |\nabla \phi_\varepsilon|^2 \geq \mathrm{cap}(\overline{D}_n, \Omega_n),$$

whence, letting $n \to \infty$,

$$\lim_n \mathrm{cap}(\overline{D}_n, \Omega_n) \leq \int |\nabla \phi_\varepsilon|^2 \quad \forall \varepsilon > 0.$$

On the other hand, by monotone convergence,

$$\int |\nabla \phi_\varepsilon|^2 = \frac{1}{(1-2\varepsilon)^2} \int_{\{x\,:\,\varepsilon\leq\phi(x)\leq 1-\varepsilon\}} |\nabla\phi|^2 \to \int_\Omega |\nabla \phi|^2 \quad \text{as } \varepsilon\to 0,$$

and we conclude that

$$\lim_n \operatorname{cap}(\overline{D}_n, \Omega_n) \leq \int |\nabla\phi|^2,$$

which in turn implies that

$$\lim_n \operatorname{cap}(\overline{D}_n, \Omega_n) \leq \operatorname{cap}(\overline{D}, \Omega).$$

□

Proposition 1.2. *Let $D \Subset \Omega$ be relatively compact domains with smooth boundaries $\overline{\partial_0 D}$ and $\overline{\partial_0\Omega}$ transversal to ∂M. Then there exists $u \in W_0(\overline{D}, \Omega) \cap C^\infty((\Omega\setminus\overline{D})\cup\partial_1(\Omega\setminus\overline{D}))$ such that $0\leq u\leq 1$ and*

$$\operatorname{cap}(\overline{D}, \Omega) = \int_\Omega |\nabla u|^2.$$

Proof. Consider the mixed boundary value problem

$$\begin{cases} \Delta u = 0 \ \text{ in } \operatorname{int}(\Omega\setminus\overline{D}) \\ \frac{\partial u}{\partial\nu} = 0 \ \text{ on } \partial_1(\Omega\setminus\overline{D}) \\ u = 0 \text{ on } \partial_0\Omega \ , u = 1 \text{ on } \partial_0 D. \end{cases} \tag{7}$$

If follows from [21], and the well known local regularity theory, that (7) has a classical solution $u\in C^0(\overline{\Omega}\setminus D)\cap C^\infty((\Omega\setminus\overline{D})\cup\partial_1(\Omega\setminus\overline{D}))$. By the strong maximum principle and the boundary point lemma, it follows that $0<u<1$ on $\Omega\setminus\overline{D}$. We extend u to Ω by setting it equal to 1 on D. To show that $u\in W^{1,2}(\Omega)$, choose $\varepsilon\in(0,1)$ such that ε and $1-\varepsilon$ are regular values of u on $\operatorname{int}(\Omega\setminus\overline{D})$ and on $\partial_1(\Omega\setminus\overline{D})$, and let $\Omega_\varepsilon = \{x : u(x)\geq\varepsilon\}$, $D_\varepsilon = \{x : u(x) > 1-\varepsilon\}$ and

$$u_\varepsilon = \frac{u-\varepsilon}{1-2\varepsilon},$$

so that $u_\varepsilon\in C^2(\overline{\Omega}_\varepsilon\setminus D_\varepsilon)$ satisfies

$$\begin{cases} \Delta u_\varepsilon = 0 \ \text{ in } \operatorname{int}(\Omega_\varepsilon\setminus\overline{D}_\varepsilon) \\ \frac{\partial u_\varepsilon}{\partial\nu} = 0 \ \text{ on } \partial_1(\Omega_\varepsilon\setminus\overline{D}_\varepsilon) \\ u_\varepsilon = 0 \text{ on } \partial_0\Omega_\varepsilon \ , u_\varepsilon = 1 \text{ on } \partial_0 D_\varepsilon, \end{cases}$$

By the usual Dirichlet principle u_ε is the equilibrium potential of the capacitor $(\overline{D}_\varepsilon, \Omega_\varepsilon)$, and, in particular,

$$\frac{1}{1-2\varepsilon}\int_{\Omega_\varepsilon\setminus D_\varepsilon} |\nabla u|^2 = \int_{\Omega_\varepsilon\setminus D_\varepsilon} |\nabla u_\varepsilon|^2 = \operatorname{cap}(\overline{D}_\varepsilon, \Omega_\varepsilon) \tag{8}$$

Indeed, let $\phi \in \mathrm{Lip}_c(\Omega_\varepsilon)$ with $\phi = 1$ on D_ε, and let $v = u_\varepsilon - \phi$. Then $\phi = u_\varepsilon - v$ and we have

$$\int_{\Omega\varepsilon} |\nabla\phi|^2 = \int_{\Omega_\varepsilon\setminus D_\varepsilon} |\nabla(u_\varepsilon - v)|^2 = \int_{\Omega_\varepsilon\setminus D_\varepsilon} (|\nabla u_\varepsilon|^2 + |\nabla v|^2 - 2\langle\nabla u_\varepsilon, \nabla v\rangle)$$

Since $\Delta u_\varepsilon = 0$ on $\Omega_\varepsilon \setminus D_\varepsilon$ and $v = 0$ on $\partial_0(\Omega_\varepsilon \setminus \overline{D}_\varepsilon)$ while $\partial u_\varepsilon/\partial\nu = 0$ on $\partial_1(\Omega_\varepsilon \setminus \overline{D}_\varepsilon)$,

$$\int_{\Omega_\varepsilon\setminus D_\varepsilon} \langle\nabla u_\varepsilon, \nabla v\rangle) = -\int_{\Omega_\varepsilon\setminus D_\varepsilon} v\Delta u_\varepsilon + \int_{\partial_0(\Omega_\varepsilon\setminus D_\varepsilon)\cup\partial_1(\Omega_\varepsilon\setminus D_\varepsilon)} \langle\nabla u_\varepsilon, \nu\rangle v = 0,$$

so that

$$\int_{\Omega\varepsilon} |\nabla\phi|^2 = \int_{\Omega_\varepsilon\setminus D_\varepsilon} (|\nabla u_\varepsilon|^2 + |\nabla v|^2) \geq \int_{\Omega_\varepsilon\setminus D_\varepsilon} |\nabla u_\varepsilon|^2, .$$

as claimed.

Letting $\varepsilon \to 0$ $\Omega_\varepsilon \setminus D_\varepsilon \nearrow \Omega \setminus D$, so that, by monotone convergence, the integral in (8) converges to

$$\int_{\Omega\setminus D} |\nabla u|^2.$$

On the other hand, by the previous lemma,

$$\mathrm{cap}(\overline{D}_\varepsilon, \Omega_\varepsilon) \to \mathrm{cap}(\overline{D}, \Omega), \quad \text{as } \varepsilon \to 0$$

and we conclude that $u \in W^{1,2}(\Omega)$ so that, in fact, $u \in W_0(\overline{D}, \Omega)$ and

$$\int_\Omega |\nabla u|^2 = \mathrm{cap}(\overline{D}, \Omega),$$

as required to complete the proof. □

Remark 1.3. It is worth to point out that the equilibrium potential u of the capacitor $(\overline{D}, \Omega)$ constructed using Lieberman approach coincides with the one obtained by applying the direct calculus of variations to the energy functional on the closed convex space

$$W^{1,2}_\Gamma\left(\Omega\setminus\overline{D}\right) = \left\{u \in W^{1,2}(\mathrm{int}\Omega) : u|_{\partial_0 D} = 0 \text{ and } u|_{\partial_0\Omega} = 1\right\}.$$

Here, Dirichlet data are understood in the trace sense. Thanks to the global $W^{1,2}$-regularity established in Proposition 1.2, this follows e.g. either from maximum principle considerations or from the convexity of the energy functional.

Proposition 1.4. *Let D be a relatively compact domain and let Ω_j be an increasing exhaustion of M by relatively compact open domains with $\overline{D} \subset \Omega_1$. Assume that $\overline{\partial_0 D}$ and $\overline{\partial_0\Omega_j}$ are smooth and transversal to ∂M, and for every j, let u_j be the equilibrium potential of the capacitor $(\overline{D}, \Omega_j)$ constructed in Proposition 1.2. Then u_j converges monotonically to a function $u \in C(M) \cap W^{1,2}_{loc}(M) \cap C^2(M \setminus \overline{D})$ such that $0 \leq u \leq 1$, $u = 1$ on $\overline{D}$, u is* *harmonic on $M \setminus \overline{D}$, $\partial u/\partial\nu = 0$ on $\partial_1(M \setminus \overline{D})$ and u is a weak Neumann supersolution of the Laplace equation on M. Moreover $\nabla u \in L^2(M)$,*

$$\mathrm{cap}(\overline{D}) = \int_M |\nabla u|^2.$$

Proof. Extend u_j to all of M by setting it equal to zero in $M \setminus \Omega_j$. It follows by the comparison principle that $0 \le u_j \le u_{j+1} \le 1$ in $\Omega_j \setminus \overline{D}$, and therefore the sequence u_j converges monotonically to a function u. Note that since $u_j(x) \le u(x) \le 1$ and $u_j(x) \to 1$ as $x \to y \in \partial_0 D$ is follows that u is continuous on $\overline{D}$ and there it is equal to 1. Moreover, by the Schauder type estimate contained in Lemma 1 in [21], for every $\alpha \in (0,1)$, every j_o and every sufficiently small $\eta > 0$ there exists a constant C depending only on α η, j_o and on the geometry of M in a neighborhood of $B_{j_o,\eta} = \{x \in \Omega_{j_o} \setminus \overline{D} : \mathrm{dist}(x, \partial_0 D \cup \partial_0 \Omega_{jo}) \ge \eta\}$ such that, for every $j \ge j_o$

$$||u_j||_{C^{2,\alpha}(B_\eta)} \le C \sup_{B_{\eta/2}} |u_j(x)|.$$

It follows immediately that (possibly passing to a subsequence) the sequence u_j converges in $C^2(B_{j_o,\eta})$ for every j_o and $\eta > 0$ so that the limit function u is harmonic in $\mathrm{int}\, M \setminus \overline{D}$ and C^2 up to $\partial_1(M \setminus \overline{D})$ where it satisfies the Neumann boundary condition $\partial u/\partial\nu = 0$. Summing up, $u \in C^0(M \setminus D) \cup C^2((M \setminus \overline{D}) \cup (\partial_1(M \setminus \overline{D})))$ is a classical solution of the mixed boundary problem

$$\begin{cases} \Delta u \ge 0 \text{ on } \mathrm{int}(M \setminus \overline{D}) \\ \frac{\partial u}{\partial \nu} \le 0 \ \text{ on } \ \partial_1(M \setminus \overline{D}) \\ u = 1 \text{ on } \partial_0 D \\ 0 \le u \le 1. \end{cases}$$

On the other hand, since

$$\int_{\Omega_j} |\nabla u_j|^2 = \mathrm{cap}(\overline{D}, \Omega_j) \searrow \mathrm{cap}(\overline{D}),$$

the sequence $u_j \in C^0_c(M) \cap W^{1,2}(\mathrm{int} M)$ converges pointwise to u and ∇u_j is bounded in $L^2(M)$. It follows easily (see, e.g., Lemma 1.33 in [13]) that $\nabla u \in L^2(M)$ and $\nabla u_j \to \nabla u$ weakly in L^2. By the weak lower semicontinuity of the energy functional, it follows that

$$\int_M |\nabla u|^2 \le \liminf_j \int_M |\nabla u_j|^2 = \mathrm{cap}(\overline{D})$$

On the other hand, By Mazur's Lemma, a convex combination $\tilde{u}_j$ of the u_j is such that $\nabla \tilde{u}_j \to \nabla u$ strongly in $L^2(M)$, and since each $\tilde{u}_j \in C^0(M) \cap W^{1,2}(\mathrm{int} M)$ is compactly supported, and equal to 1 on $\overline{D}$, it is admissible for the capacitor $(\overline{D}, M)$ and we deduce that

$$\int_M |\nabla u|^2 = \lim \int_M |\nabla \tilde{u}_j|^2 \ge \mathrm{cap}(\overline{D}),$$

and we conclude that

$$\int_M |\nabla u|^2 = \operatorname{cap}(\overline{D}),$$

as required.

Finally, assume that u is non-constant so that, by the strong maximum principle, $u < 1$ in $M \setminus \overline{D}$. Let $\eta_n \to 1$ be a sequence of regular values of $u_{|\mathrm{int}(M\setminus\overline{D})}$ and $u_{\partial_1(M\setminus\overline{D})}$, and set $\Gamma_n = \{x : u(x) < \eta_n\}$. Using the fact that $\Delta u = 0$ on $\Gamma_n \subset M \setminus \overline{D}$, $\partial u/\partial\nu = 0$ on $\partial_1\Gamma_n$ and $\partial u/\partial\nu \geq 0$ on $\partial_0\Gamma_n$, given $0 \leq \rho \in C_c^\infty(M)$, we compute

$$\int_M \langle \nabla u, \nabla \rho\rangle = \lim_n \int_{\Gamma_n} \langle \nabla u, \nabla \rho\rangle = \lim_n \{-\int_{\Gamma_n} \rho\Delta u + \int_{\partial_0\Gamma_n \cup \partial_1\Gamma_n} \rho\langle \nabla u, \nu\rangle\} \geq 0,$$

and u is a weak Neumann supersolution of the Laplace equation on M. □

We then obtain the announced equivalent characterization of parabolicity for a manifold with non-empty boundary.

Theorem 1.5. *Let $(M, \langle\,,\rangle)$ be an oriented, connected Riemannian manifold with boundary $\partial M \neq \emptyset$. The following are equivalent:*

(i) *The capacity of every compact set K in M is zero.*

(ii) *For every relatively compact open domain $D \Subset M$ there exists an increasing sequence of functions $u_j \in C_c^0(M) \cap W^{1,2}(\mathrm{int}M)$ with $u_j = 1$ on D, $0 \leq u_j \leq u_{j+1} \leq 1$, u_j harmonic in the set $\{x : 0 < u_j(x) < 1\} \cap \mathrm{int}M$, such that*

$$\int_M |\nabla u_j|^2 \to 0 \qquad \text{as } j \to +\infty.$$

(iii) *M is parabolic.*

Proof. (i) ⇒ (ii). Assume first that $\operatorname{cap}(K) = 0$ for every compact set K in M, let D be as in (ii) and let $\{\Omega_j\}$ be an increasing exhastion of M by relatively compact open set with smooth boundary transversal to ∂M with $\overline{D} \subset \Omega_1$. For every j let u_j be the equilibrium potential of the capacitor $(\overline{D}, \Omega_j)$, and extend u_j to be 0 off Ω_j. Then u_j has the regularity properties listed in (ii), and, by Proposition 1.2,

$$\int |\nabla u_j|^2 = \operatorname{cap}(\overline{D}, \Omega_j) \to \operatorname{cap}(\overline{D}) = 0.$$

(ii) ⇒ (i) Conversely, assume that (ii) holds. Clearly it suffices to prove that $\operatorname{cap}(\overline{D}) = 0$ for every relatively compact open domain D with smooth boundary transversal to ∂M. Choose an increasing exhaustion of M by relatively compact domains Ω_j with smooth boundary transversal to ∂M such that $\mathrm{supp} u_j \Subset \Omega_j$. Then

$$\operatorname{cap}(\overline{D}) = \lim_j \operatorname{cap}(\overline{D}, \Omega_j) \leq \lim_j \int_{\Omega_j} |\nabla u_j|^2 \to 0,$$

as required.

(i) ⇒ (iii) Suppose that $\mathrm{cap}(K) = 0$ for every compact set in M, and let $u \in C^0(M) \cap W^{1,2}_{loc}(M)$ satisfy, in the weak Neumann sense,

$$\begin{cases} \Delta u \geq 0 \text{ on } \mathrm{int}M \\ \frac{\partial u}{\partial \nu} \leq 0 \text{ on } \partial M \\ \sup_M u < +\infty. \end{cases} \tag{9}$$

Let $v = \sup_M u - u + 1$, so that $v \geq 1$ and, by definition of weak solution of the differential problem (9), v satisfies

$$\int \langle \nabla v, \nabla \rho \rangle \geq 0 \quad \forall 0 \leq \rho \in C^0(M) \cap W^{1,2}_0(M).$$

Next, for every relatively compact domain D, let $\varphi \in \mathrm{Lip}_c(M)$ with $\varphi = 1$ on D, and $0 \leq \varphi \leq 1$. Using $\rho = \varphi^2 v^{-1} \in C^0(M) \cap W^{1,2}_c(M)$ as a test function we have

$$\begin{aligned} 0 \leq \int \langle \nabla v, \nabla \rho \rangle &= 2 \int \varphi \langle v^{-1} \nabla v, \nabla \varphi \rangle - \int \varphi^2 |v^{-1} \nabla v|^2 \\ &\leq 2 \int \varphi |v^{-1} \nabla v| |\nabla \varphi| - \int \varphi^2 |v^{-1} \nabla v|^2. \end{aligned}$$

Rearranging, using Young's inequality $2ab \leq 2a^2 + \frac{1}{2}b^2$, and recalling that $\varphi = 1$ on D we obtain

$$\int_D |v^{-1} \nabla v|^2 \leq 4 \int |\nabla \varphi|^2,$$

and taking the inf of the right hand side over all Lip_c function φ which are equal to 1 on D we conclude that

$$\int_D |v^{-1} \nabla v|^2 \leq 4\mathrm{cap}(\overline{D}) = 0$$

Thus v and therefore u is constant on every relatively compact domain D. Thus u is constant on M, and M is parabolic in the sense of Definition 0.5.

(iii) ⇒ (i) Assume by contradiction that there exists compact set K with nonzero capacity. Without loss of generality we can suppose that K is the closure of a relatively compact open domain D with smooth boundary $\partial_0 D$ transversal to ∂M. Let u be the equilibrium potential of $\overline{D}$ constructed in Proposition 1.4, which is non-constant since the capacity of $\overline{D}$ is positive. But then $u \in C^0(M) \cap W^{1,2}(\mathrm{int}M)$ is a non-constant bounded weak Neumann superharmonic function, contradicting the assumed parabolicity of M. □

2. Maximum principles & height estimates

It is a classical result by L.V. Ahlfors that a Riemannian manifold N (without boundary) is parabolic if and only if, for every domain $D \subseteq N$ with $\partial D \neq \emptyset$ and for every bounded above, subharmonic function u on D it holds that $\sup_D u = \sup_{\partial D} u$. The result has been extended in the setting of p-parabolicity in [29]. This section aims to provide a new form of the

Ahlfors characterization which is valid on manifolds with boundary. This, in turn, will be used to obtain estimate of the height function of complete hypersurfaces with constant mean curvature (CMC for short) immersed into product spaces of the form $N \times \mathbb{R}$.

2.1. **Global maximum principles.** We are going to prove the Ahlfors-type characterization of parabolicity stated in Theorem 0.9. Actually, a version of this global maximum principle involving the whole manifold and without any Neumann condition will be crucial in the geometric applications. This is the content of Theorem 0.10 that will be proved at the end of the section. Throughout the present section we will abundantly use the fact that truncating from below a weak Neumann subsolution still yields a weak Neumann subsolution. The proof, that should be well known, proceeds exactly as in the case without boundary and will be recalled in Appendix A.

Proof (of Theorem 0.9). Assume first that M is parabolic and suppose, by contradiction, that there exists a domain $D \subseteq M$ and a function u as in the statement of the Theorem, such that

$$\sup_D u > \sup_{\partial_0 D} u.$$

Let $\varepsilon > 0$ be so small that

$$\sup_D u > \sup_{\partial_0 D} u + \varepsilon.$$

Then, the open set $D_\varepsilon = \{x \in D : u > \sup_D u - \varepsilon\} \neq \emptyset$ satisfies $\overline{D}_\varepsilon \subset D$ and, therefore,

$$u_\varepsilon = \begin{cases} \max\{u, \sup_D u - \varepsilon\} & \text{on } D \\ \sup_D u - \varepsilon & \text{on } M \backslash D \end{cases}$$

well defines a $C^0(M) \cap W^{1,2}_{loc}(M)$-subsolution of the Laplace equation on M. Furthermore, $\sup_M u_\varepsilon = \sup_D u < +\infty$. It follows from the very definition of parabolicity that u_ε is constant on M. In particular, if we suppose to have chosen $\varepsilon > 0$ in such a way that $\sup_D u - \varepsilon$ is not a local maximum for u, then $u_\varepsilon = \sup_D u - \varepsilon$ on $\partial D_\varepsilon \neq \emptyset$ and we conclude

$$u \equiv \sup_D u - \varepsilon, \text{ on } D,$$

which is absurd.

Suppose now that, for every domain $D \subseteq M$ with $\partial_0 D \neq \emptyset$ and for every $u \in C^0(\overline{D}) \cap W^{1,2}_{loc}(D)$ satisfying

$$\begin{cases} \Delta u \geq 0 & \text{on } \operatorname{int} D \\ \dfrac{\partial u}{\partial \nu} \leq 0 & \text{on } \partial_1 D \\ \sup_D u < +\infty & \end{cases}$$

in the weak sense, it holds

$$\sup_D u = \sup_{\partial_0 D} u.$$

By contradiction assume that M is not parabolic. Then, there exists a non-constant function $v \in C^0(M) \cap W^{1,2}_{loc}(M)$ satisfying

$$\begin{cases} \Delta v \geq 0 & \text{on int}M \\ \dfrac{\partial v}{\partial \nu} \leq 0 & \text{on } \partial M \\ v^* = \sup_M v < +\infty. \end{cases}$$

Given $\eta < v^*$ consider the domain $\Omega_\eta = \{x \in M : v(x) > \eta\} \neq \emptyset$. We can choose η sufficiently close to v^* in such a way that $\text{int}M \not\subseteq \Omega_\eta$. In particular, $\partial\Omega_\eta \subseteq \{v = \eta\}$ and $\partial_0\Omega_\eta \neq \emptyset$. Now, $v \in C^0\left(\overline{\Omega}_\eta\right) \cap W^{1,2}_{loc}(\Omega_\eta)$ is a bounded above weak Neumann subsolution on Ω_η. Moreover,

$$\sup_{\partial_0\Omega_\eta} v = \eta < \sup_{\Omega_\eta} v,$$

contradicting our assumptions. □

If we take $D = M$ in the first half of the above proof it appears that the Neumann condition plays no role. Indeed:

Proof (of Theorem 0.10). If, by contradiction,

$$\sup_M u > \sup_{\partial M} u$$

then, we can choose $\varepsilon > 0$ so small that

$$\sup_M u > \sup_{\partial M} u + 2\varepsilon.$$

Define $u_\varepsilon \in C^0(M) \cap W^{1,2}_{loc}(M)$ by setting

$$u_\varepsilon = \begin{cases} \max\left(u, \sup_M u - \varepsilon\right) & \text{on } \Omega_{2\varepsilon} \\ \sup_M u - \varepsilon & \text{on } M \backslash \Omega_{2\varepsilon}, \end{cases}$$

where we have set

$$\Omega_{2\varepsilon} = \left\{ x \in M : u(x) > \sup_M u - 2\varepsilon \right\}.$$

Since $\overline{\Omega}_{2\varepsilon} \subset \text{int}M$, we have that u_ε is constant in a neighborhood W of ∂M. Since $\Delta u \geq 0$ weakly on $\text{int}M$, it follows that u_ε is a weak Neumann subsolution on M. Indeed, by assumption and by the fact that the maximum of subharmonic functions is still subharmonic we have $\Delta u_\varepsilon \geq 0$ weakly on $\text{int}M$. Now, let $0 \leq \varphi \in C^\infty_c(M)$. On noting that u_ε is constant on $W \cap \text{supp}(\varphi)$, we find a cut-off function $0 \leq \eta \in C^\infty_c(\text{int}M)$ such that

$$\langle \nabla u_\varepsilon, \nabla\varphi \rangle = \langle \nabla u_\varepsilon, \nabla(\eta\varphi) \rangle.$$

Simply consider the compact set $K = (M \setminus W) \cap \mathrm{supp}(\varphi)$ of $\mathrm{int}M$, take a neighborhood $\Omega \Subset \mathrm{int}(M)$ of K and choose $0 \le \eta \le 1$ such that $\eta = 1$ in Ω and $\mathrm{supp}(\eta) \subset \mathrm{int}M$. Thus

$$-\int_M \langle \nabla u_\varepsilon, \nabla \varphi \rangle = -\int_{\mathrm{int}M} \langle \nabla u_\varepsilon, \nabla(\eta\varphi) \rangle \ge 0,$$

as claimed.

Moreover, $\sup_M u_\varepsilon = \sup_M u < +\infty$ so that, by parabolicity, $u_\varepsilon \equiv \sup_M u - \varepsilon$, a contradiction. □

2.2. **Height estimates for CMC hypersurfaces in product spaces.** We now present some applications of this global maximum principle to get height estimates both for H-hypersurfaces with boundary in product spaces and for H-graphs over manifolds with boundary. By an H-hypersurface of $N \times \mathbb{R}$ we mean and oriented hypersurface Σ with constant mean curvature H with respect to a choice of its Gauss map. An H-graph over the m-dimensional Riemannian manifold M with boundary $\partial M \neq \emptyset$ is an embedded H-hypersurfaces given by $\Sigma = \Gamma_u(M)$ where $\Gamma_u : M \to M \times \mathbb{R}$ is defined, as usual, by $\Gamma_u(x) = (x, u(x))$, for some smooth function $u : M \to \mathbb{R}$. Sometimes, we will also allow that $u \in C^\infty(\mathrm{int}(M)) \cap C^0(M)$ and, in this case, we will speak of a *proper H-graph.* The downward (pointing) unit normal to Σ is defined by

$$\mathcal{N} = \frac{1}{\sqrt{1 + |\nabla_M u|^2}} (\nabla_M u, -1).$$

With respect to $\mathcal{N}$, the mean curvature of the smooth graph Σ writes as

$$H = -\frac{1}{m} \mathrm{div}_M \left(\frac{\nabla_M u}{\sqrt{1 + |\nabla_M u|^2}} \right).$$

On the other hand, let M_Σ be the original manifold M endowed with the metric pulled back from $M \times \mathbb{R}$ via Γ_u, so that M_Σ is isometric to the hypersurface Σ of $M \times \mathbb{R}$ with its induced metric. It is well known that the mean curvature vector field of the isometric immersion Γ_u

$$\mathbf{H}(x) = H(x) \mathcal{N}(x)$$

satisfies

$$\Delta_\Sigma \Gamma_u = m\mathbf{H},$$

where Δ_Σ denotes the Laplacian on manifold-valued maps. Since Δ_Σ is linear with respect to the Riemannian product structure in the codomain, from the above we also get

$$\begin{aligned}\Delta_{\Sigma} u &= \frac{1}{\sqrt{1+|\nabla_M u|^2}} \operatorname{div}_M \left(\frac{\nabla_M u}{\sqrt{1+|\nabla_M u|^2}} \right) \\ &= -\frac{m}{\sqrt{1+|\nabla_M u|^2}} H(x)\end{aligned} \tag{10}$$

With this preparation, we begin by noting the following version of Lemma 1 in [20].

Lemma 2.1. *Let $(N, \langle\, , \rangle)$ be an m-dimensional complete manifold without boundary and let $M \subset N$ be a closed domain with smooth boundary $\partial M \neq \emptyset$. Consider a smooth graph $\Sigma = \Gamma_u(M) \subset N \times \mathbb{R}$ over M with smooth boundary*

$$\partial \Sigma \subset N \times \{0\}.$$

Assume that

$$\sup_M |u| + \sup_M |H| < +\infty. \tag{11}$$

Then there exists a constant $C = C(m, \sup_M |u|, \sup_M |H|) > 0$ such that, for every $\delta > 0$ and $R > 1$,

$$\operatorname{vol} B_R^{\Sigma}(\bar{p}) \leq C \left(1 + \frac{1}{\delta R} \right) \operatorname{vol} \left(M \cap B_{(1+\delta)R}^N (\bar{x}) \right),$$

where $\bar{x}$ is a reference point in N and $\bar{p} = (\bar{x}, u(\bar{x}))$. Moreover, the following estimate

$$\operatorname{vol} B_R^{\Sigma}(\bar{p}) \leq C \left\{ \operatorname{vol} B_R^N(\bar{x}) + \operatorname{Area}\left(\partial B_R^N(\bar{x}) \right) \right\}$$

holds for almost every $R > 1$.

Proof. Note that

$$\begin{aligned} d_{\Sigma}\left((\bar{x}, u(\bar{x})), (x, u(x)) \right) &\geq d_{N\times\mathbb{R}}\left((\bar{x}, u(\bar{x})), (x, u(x)) \right) \\ &\geq \max\left\{ d_N(\bar{x}, x), |u(\bar{x}) - u(x)| \right\}. \end{aligned}$$

Therefore, letting $\bar{p} = (\bar{x}, u(\bar{x}))$, we have

$$\begin{aligned} B_R^{\Sigma}(\bar{p}) &\subseteq \Sigma \cap B_R^{N\times\mathbb{R}}(\bar{p}) \\ &\subseteq \left(M \cap B_R^N(\bar{x}) \right) \times \left(-R + u(\bar{x}), R + u(\bar{x}) \right). \end{aligned}$$

If we denote by $\Pi_N : \Sigma \to N$ the projection on the N factor, it follows that

$$
\begin{aligned}
\mathrm{vol} B^{\Sigma}_R(\bar p) &= \int_{\Pi_N\left(B^{\Sigma}_R(\bar p)\right)} \sqrt{1+|\nabla u|^2}\, d\mathrm{vol}_N \\
&\le \int_{\Omega_R} \sqrt{1+|\nabla u|^2}\, d\mathrm{vol}_N \\
&= \int_{\Omega_R} \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} d\mathrm{vol}_N + \int_{\Omega_R} \frac{1}{\sqrt{1+|\nabla u|^2}} d\mathrm{vol}_N \\
&\le \int_{\Omega_R} \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} d\mathrm{vol}_N + \mathrm{vol}(M \cap B^N_R(\bar x)),
\end{aligned} \tag{12}
$$

where $\Omega_R = \{x \in M \cap B^N_R(\bar x) \,:\, |u(x) - u(\bar x)| < R\} \subseteq M \cap B^N_R(\bar x)$. Now, for any $\delta > 0$, we choose a cut-off function ρ as follows:

$$
\rho(x) = \begin{cases} 1 & \text{on} \quad B_R(\bar x) \\ \frac{(1+\delta)R - r(x)}{\delta R} & \text{on} \quad B_{(1+\delta)R}(\bar x) \backslash B_R(\bar x) \\ 0 & \text{elsewhere,} \end{cases}
$$

where $r(x)$ denotes the distance function on N from the reference point $\bar x$. Since

$$
X = \rho u \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}
$$

is a compactly supported vector field that vanishes on ∂M and on $\partial B^N_{(1+\delta)R}(\bar x)$, as an application of the divergence theorem we get

$$
\begin{aligned}
0 =& \int_{M\cap B^N_{(1+\delta)R}(\bar x)} \mathrm{div}(X) d\mathrm{vol}_N \\
=& - m \int_{M\cap B^N_{(1+\delta)R}(\bar x)} \rho H u d\mathrm{vol}_N + \int_{M\cap B^N_{(1+\delta)R}(\bar x)} \frac{\rho\, |\nabla u|^2}{\sqrt{1+|\nabla u|^2}} d\mathrm{vol}_N \\
& - \frac{1}{\delta R} \int_{M\cap (B^N_{(1+\delta)R}(\bar x) \backslash B^N_R(\bar x))} u \frac{\langle \nabla u, \nabla r\rangle}{\sqrt{1+|\nabla u|^2}} d\mathrm{vol}_N .
\end{aligned}
$$

Hence

$$\begin{aligned}\int_{\Omega_R} \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} d\mathrm{vol}_N \leq & \int_{M\cap B^N_R(\bar{x})} \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} d\mathrm{vol}_N \\ \leq & \int_{M\cap B^N_{(1+\delta)R}(\bar{x})} \frac{\rho|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} d\mathrm{vol}_N \\ \leq & m \sup_M |u| \sup_M |H| \mathrm{vol}(M\cap B^N_{(1+\delta)R}(\bar{x})) \\ & + \frac{\sup_M |u|}{\delta R} \mathrm{vol}(M\cap (B^N_{(1+\delta)R}(\bar{x}) \setminus B^N_R(\bar{x}))).\end{aligned}$$

Inserting this latter into (12) gives, for every $R > 1$,

$$\begin{aligned}\mathrm{vol} B^\Sigma_R(\bar{p}) \leq & \frac{C}{2} \Big\{ \mathrm{vol}(M\cap B^N_R(\bar{x})) + \mathrm{vol}(M\cap B^N_{(1+\delta)R}(\bar{x})) \\ & + \frac{1}{\delta R} \mathrm{vol}(M\cap (B^N_{(1+\delta)R}(\bar{x}) \setminus B^N_R(\bar{x}))) \Big\}.\end{aligned}$$

To conclude, we let $\delta \to 0$ and we use the co-area formula. □

Remark 2.2. We note that, actually, the somewhat weaker conclusions

$$\mathrm{vol} B^\Sigma_R(\bar{p}) \leq C\left(1+\frac{1}{\delta}\right) \mathrm{vol}\left(M\cap B^N_{(1+\delta)R}(\bar{x})\right), \tag{13}$$

and

$$\mathrm{vol} B^\Sigma_R(\bar{p}) \leq C\left\{\mathrm{vol} B^N_R(\bar{x}) + R\,\mathrm{Area}\left(\partial B^N_R(\bar{x})\right)\right\} \tag{14}$$

hold under the assumption

$$\sup_M |uH| < +\infty.$$

Indeed, to overcome the problem that u can be unbounded, following the proof in the minimal case $H \equiv 0$, one can apply the divergence theorem to the vector field

$$X = \rho u_R \frac{\nabla u}{\sqrt{1+|\nabla u|^2}},$$

where u_R is defined as

$$u_R = \begin{cases} u(\bar{x}) - R & \text{if} \quad u(x) \leq u(\bar{x}) - R \\ u(x) & \text{if} \quad |u(x) - u(\bar{x})| < R \\ u(\bar{x}) + R & \text{if} \quad u(x) \geq u(\bar{x}) + R, \end{cases}$$

for all R such that $u(\bar{x}) - R < 0$, and note that by definition

$$\int_{\Omega_R} \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} d\mathrm{vol}_N = \int_{M\cap B^N_R(\bar{x})} \frac{|\nabla u_R|^2}{\sqrt{1+|\nabla u|^2}} d\mathrm{vol}_N$$

We also note that similar conclusions continue to hold if the condition that u vanishes on ∂M is replaced by the assumption that $\langle \mathcal{N}, \mathcal{N}_o \rangle$ has constant sign on $\partial\Sigma$, where $\mathcal{N}_0 = (-\nu, 0)$ is the inward unit normal to the cylinder $\partial M \times \mathbb{R}$, and we assume either (11) is valid, or that $H = 0$, i.e., the graph is minimal. Indeed, the assumption amounts to $\langle \nabla u, \nu \rangle$ having constant sign. If (11) holds, then the conclusion of the Lemma is obtained applying the divergence theorem to the vector field

$$X = \rho(u + c \sup |u|) \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}},$$

with $c = -\mathrm{sgn}\langle \nabla u, \nu \rangle$, and noting that, by definition, the boundary term is negative. If $H = 0$, one uses instead the vector field

$$X = \rho[u_R + c(|u(\bar{x})| + R)] \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}},$$

to obtain (13) and (14).

Remark 2.3. It could be interesting to observe that, in certain situations, an improved version of Lemma 2.1 can be obtained from the a-priori gradient estimates due to N. Koreevar, X.-J. Wang and J. Spruck, [16, 35, 34]. See also [32] where the injectivity radius assumption has been removed. More precisely, we have the next simple result. We explicitly note that, with respect to Lemma 2.1, no assumption on $\partial\Sigma$ is required. Moreover, the volume estimate involves the same radius $R > 0$ without any further contribution.

Lemma 2.4. *Let $(N, \langle\, , \rangle)$ be a complete, m-dimensional Riemannian manifold (without boundary) satisfying $Sec_N \geq -K$ and let $M \subset N$ be a closed domain with smooth boundary $\partial M \neq \emptyset$. Suppose we are given a vertically bounded graph $\Sigma_\varepsilon = \Gamma_u(\mathcal{U}_\varepsilon(M))$ with bounded mean curvature H, parametrized over an ε-neighborhood $\mathcal{U}_\varepsilon(M)$ of M. Let $\Sigma = \Gamma_u(M)$. Then, there exists a constant $C = C(m, \varepsilon, H, K, \sup_M |u|, \sup_M |H|) > 0$ such that*

$$\mathrm{vol} B_R^\Sigma(\bar{p}) \leq C \mathrm{vol}\left(M \cap B_R^N(\bar{x})\right),$$

for every $R > 0$, where $\bar{x} \in \mathrm{int} M$ is a reference point and $\bar{p} = (\bar{x}, u(\bar{x}))$.

Proof. Indeed, since

$$\mathrm{vol} B_R^\Sigma(\bar{p}) = \int_{\Pi_N\left(B_R^\Sigma(\bar{p})\right)} \sqrt{1 + |\nabla u|^2} d\mathrm{vol}_N \leq \int_{M \cap B_R^N(\bar{x})} \sqrt{1 + |\nabla u|^2} d\mathrm{vol}_N,$$

we have only to show that $|\nabla u|$ is uniformly bounded on M. To this end, note that $u : \mathcal{U}_\varepsilon(M) \to \mathbb{R}$ is a bounded function defining a bounded mean curvature graph $\Gamma_u(\mathcal{U}_\varepsilon(M))$. Therefore, we can apply Theorem 1.1 in [34] to either $w(x) = \sup_M u - u(x) \geq 0$ or $w(x) = u(x) - \inf_M u \geq 0$ and obtain that, in fact, $\left|\nabla^M u\right|$ is uniformly bounded on every ball $B_{\varepsilon/2}^N(x) \subset \mathcal{U}_\varepsilon(M)$, with $x \in M$. This completes the proof. $\square$

Lemma 2.1 allows to prove Theorem 0.12 stated in the Introduction.

Proof (of Theorem 0.12). Without loss of generality, we can assume that $\Sigma = \{(x, u(x)) : x \in M\}$ is a (graphical) hypersurface with smooth boundary. Indeed, if we merely have $u \in C^{\infty}(\text{int}M) \cap C^0(M)$, we can always consider the smooth $H(>0)$-graph $\Sigma_\varepsilon = \{(x, u_\varepsilon(x) : x \in M_\varepsilon\}$ where, according to Sard theorem, $0 < \varepsilon << 1$ is a regular value of $u|_{\text{int}M}$ and

$$M_\varepsilon = \{x \in M \,:\, u(x) \geq \varepsilon\} \subset \text{int}M, \qquad u_\varepsilon\,(x) = u\,(x) - \varepsilon.$$

Then, using the height estimate in the smooth case we get

$$u_\varepsilon \leq \frac{1}{H} \text{ i.e. } u \leq \frac{1}{H} + \varepsilon$$

and the desired conclusion follows by letting $\varepsilon \to 0$ along a suitable sequence. The estimate $u \geq 0$ is obtained in a similar way.

Thus, from now on, Σ is a smooth graphical hypersurface with smooth boundary $\partial\Sigma$. Observe first that Σ is complete. Indeed, Σ is a closed subset of the complete manifold $N \times \mathbb{R}$, thus from the relation

$$\overline{B}^{\Sigma}_R(\overline{x}, u(\overline{x})) \subseteq \Sigma \cap (\overline{B}^{N}_R(\overline{x}) \times [u(\overline{x}) - R, u(\overline{x}) + R])$$

it follows that intrinsic closed balls in Σ are compact. Moreover, according to Lemma 2.1, since N has quadratic volume growth, so has Σ. In particular, by Theorem 0.7, if we denote by M_Σ the original domain M endowed with the metric pulled back from Σ via Γ_u, we conclude that M_Σ is parabolic. Consider now the real-valued function $w \in C^{\infty}\,(M_\Sigma)$ defined by

$$w\,(x) = Hu\,(x) - \frac{1}{\sqrt{1 + |\nabla u\,(x)|^2}}.$$

Since $Ric_N \geq 0$, it is well known that w is subharmonic; see e.g. [1]. Moreover, $w \leq 0$ on ∂M_Σ and $\sup_{M_\Sigma} w \leq H \sup_{M_\Sigma} u < +\infty$. It follows from Theorem 0.10 that

$$\sup_{M_\Sigma} w = \sup_{\partial M_\Sigma} w \leq 0$$

and, therefore,

$$H \sup_{M_\Sigma} u - 1 \leq \sup_{M_\Sigma} w \leq 0.$$

This shows that $u \leq 1/H$. To conclude the proof, observe that, by (10), $u \in C^{\infty}\,(M_\Sigma)$ is a superharmonic function. Moreover, by assumption, u is bounded and $u = 0$ on ∂M_Σ. Therefore, using again Theorem 0.10 in the form of a minimum principle, we deduce

$$\inf_{M_\Sigma} u = \inf_{\partial M_\Sigma} u = 0,$$

proving that $u \geq 0$. □

Remark 2.5. It is well known that, in case $\partial M = \emptyset$, the above volume growth assumption implies that the vertically bounded H-graph must be necessarily minimal, $H = 0$. Actually, according to Theorem 5.1 in [30], the same conclusion holds if $\text{vol}B_R \leq C_1 e^{C_2 R^2}$ for some constants $C_1, C_2 > 0$.

Indeed, under this condition, the weak maximum/minimum principle at infinity for the mean-curvature operator holds on M. Therefore, there exists a sequence x_k along which

$$(a) \quad u(x_k) < \inf_M u + 1/k$$
$$(b) \quad mH \equiv -\operatorname{div}((1 + |\nabla_M u|^2 (x_k))^{-1/2} \nabla_M u(x_k)) < 1/k.$$

This shows that $H \le 0$. In a similar fashion we obtain the opposite inequality, proving that $H \equiv 0$. The same conclusion was also obtained in [27] by different methods.

On the other hand, if $\partial M = \emptyset$ and the volume growth of M is subquadratic then M is parabolic with respect to the mean curvature operator, [30]. Therefore, not only the H-graph is minimal, but it must be a slice of $M \times \mathbb{R}$.

Remark 2.6. Theorem 0.12 goes in the direction of generalizing Theorem 6 in [31] by A. Ros and H. Rosenberg to non-homogeneous domains. Indeed, assume that $m = 2, 3, 4$ and $\mathrm{Sec}_N \ge 0$. Then, for every $|H| > 0$, an H-graph $\Sigma = \Gamma_u(M)$ in $N \times \mathbb{R}$ over a domain $M \subseteq N$, is necessarily bounded; [31, 3, 8]. Furthermore, in case $m = 2$, it follows by the Bishop-Gromov comparison theorem that, if $\mathrm{Sec}_N \ge 0$, then N has quadratic volume growth, that is

$$\mathrm{vol} B_R^N(\bar{x}) \le \omega_2 R^2,$$

where ω_2 denotes the area of the unit ball in $\mathbb{R}^2$.

In light of the considerations above, Corollary 0.14 is now straightforward.

We end this section, by considering the more general case of an oriented CMC hypersurface in the Riemannian product $N \times \mathbb{R}$. Abstracting from the previous arguments, and up to using more involved computations as in [1], we easily obtain the proof of Theorem 0.11 stated in the Introduction.

Proof (of Theorem 0.11). Let $f : \Sigma^m \to N^m \times \mathbb{R}$ be a complete, oriented H-hypersurface isometrically immersed in $N \times \mathbb{R}$, and denote by h the projection of the image of Σ on $\mathbb{R}$ under the immersion, that is, $h = \pi_{\mathbb{R}} \circ f$. Note that

$$\Delta_\Sigma h = m \cos \Theta H \le 0, \tag{15}$$

where, we recall, $\Theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ stands for the angle between the Gauss map $\mathcal{N}$ and the vertical vector field $\partial/\partial t$. Since, by Theorem 0.7, Σ is parabolic and h is a bounded below superharmonic function, we can apply the Ahlfors maximum principle to get

$$h \ge \inf_\Sigma h = \inf_{\partial\Sigma} h = 0.$$

Consider now the function φ defined as

$$\varphi = Hh + \cos\Theta.$$

We know by Theorem 3.1 in [1] that φ is subharmonic. Since it is also bounded, applying again the Ahlfors maximum principle we conclude that

$$Hh - 1 \leq \varphi \leq \sup_{\Sigma} \varphi = \sup_{\partial\Sigma} \varphi \leq 0.$$

We have thus shown that

$$0 \leq \pi_{\mathbb{R}} \circ f(x) \leq \frac{1}{H},$$

as required. □

3. Different notions of parabolicity & remarks on minimal graphs

In this section we survey different concepts of parabolicity that can be found in the literature and establish some relations between them. We also show how the Ahlfors-maximum principle viewpoint can be used to deduce results on minimal graphs in $\mathbb{R}^3$.

Let $(M, \langle\, , \rangle)$ be a Riemannian manifold without boundary. Then, from the stochastic viewpoint, M is called parabolic if Brownian motion X_t on M is *recurrent*, that is X_t enters infinitely many times a fixed compact set with probability 1. As recorded in the survey paper [11], the recurrence of the Brownian motion for manifolds without boundary can be characterized in terms of fundamental solutions to the Laplace equation, maximum principles for superharmonic functions, capacities, heat kernel, Liouville properties for certain Schrödinger equations, volume growth conditions, function theoretic tests (Khas'minskii criterion), L^2-Stokes theorems (Kelvin-Nevanlinna-Royden criterion) and many other geometric and potential-theoretic properties.

If M has non-empty boundary $\partial M \neq \emptyset$, a quick check of the literature shows that there are many (non-equivalent) definitions of parabolicity. The most classical one, which is also the one we have adopted throughout the paper, was systematically used by A. Grigor'yan starting from [9, 10], and states that M is parabolic provided the reflected Brownian motion on M is recurrent. This is equivalent to requiring the Liouville-type property stated in Definition 0.5 above, which imposes Neumann-type boundary conditions on relevant functions. For obvious reasons, throughout this section we will refer to this property as *Neumann-parabolicity* or, shortly, *$\mathcal{N}$-parabolicity*.

Most of the geometric and functional-analytic characterizations of $\mathcal{N}$-parabolicity of manifolds without boundary have already been extended to the reflected Brownian motion; see [9, 10, 11]. Two remarkable exceptions were represented by the L^2-Stokes theorem and the Ahlfors-type maximum principles, which are some of the main topics of the present paper.

A second interesting definition can be found in a paper by R. F. De Lima, [7], who was interested in maximum principles at infinity for CMC surfaces. His definition goes in the direction of the classical Ahlfors maximum principle characterization of parabolic manifolds without boundary. Apparently there was no further research in this direction. Moreover, note that, a priori, there is no obvious relation between his notion and the behaviour of Brownian motion on M. Anyway, in the terminology of De Lima, we have the following

Definition 3.1. *A Riemannian manifold $(M, \langle\, ,\rangle)$ with boundary $\partial M \neq \emptyset$ is $\mathcal{A}$-parabolic if for every $C^0(M) \cap W^{1,2}_{\text{loc}}(\text{int}M)$ solution of the problem*

$$\begin{cases} \Delta u \geq 0 & \text{on } \text{int}M \\ \sup_M u < +\infty \end{cases}$$

it holds

$$\sup_M u = \sup_{\partial M} u.$$

As we already observed in Section 2.1, it is not difficult to prove that the classical (i.e. Neumann) definition of parabolicity implies the one introduced by De Lima. Namely,

Proposition 3.2 (=Theorem 0.10)**.** *Assume that $(M, \langle\, ,\rangle)$ is an $\mathcal{N}$-parabolic manifold with boundary $\partial M \neq \emptyset$ and let u be a weak solution of the problem*

$$\begin{cases} \Delta u \geq 0 & \text{on } \text{int}M \\ \sup_M u < +\infty. \end{cases}$$

Then

$$\sup_M u = \sup_{\partial M} u.$$

In particular, an $\mathcal{N}$-parabolic manifold with boundary is $\mathcal{A}$-parabolic

Finally, a third fruitful definition comes from very recent works in the theory of minimal surfaces in the Euclidean space, [6, 26, 22, 24]. ¿From the Brownian motion viewpoint, it states that M is parabolic provided the absorbed Brownian motion is recurrent, i.e., with probability 1 the particle reaches the boundary (and dies) in a finite time. From a deterministic viewpoint, this definition involves Dirichlet boundary conditions on the relevant functions. In this context, a Riemannian manifold is said to be parabolic if bounded harmonic functions are determined by their boundary values. This is equivalent to the following

Definition 3.3. *A Riemannian manifold M with boundary $\partial M \neq \emptyset$ is $\mathcal{D}$-parabolic if the unique solution of the problem*

$$\begin{cases} \Delta u = 0 & \text{on } \text{int}M \\ u = 0 & \text{on } \partial M \\ \sup_M |u| < +\infty \end{cases}$$

is the constant function $u \equiv 0$.

This notion of parabolicity has been used in the theory of minimal surfaces in $\mathbb{R}^3$ because it turned out to be a powerful tool in order to face the problem of determining which conformal structures are allowed on a minimal surface subjected to some geometric restrictions on its image.

The notion of $\mathcal{D}$-parabolicity is related to the classical Neumann one via the Ahlfors maximum principle. Indeed, the following result follows by applying the definition of $\mathcal{A}$-parabolicity to u and to $-u$, and Proposition 3.2.

Proposition 3.4. *Assume that $(M, \langle\, ,\rangle)$ is an $\mathcal{A}$-parabolic manifold with boundary $\partial M \neq \emptyset$ and let u be a solution of the problem*

$$\begin{cases} \Delta u = 0 & \text{on int}M \\ u = 0 & \text{on } \partial M \\ \sup_M |u| < +\infty \end{cases}$$

Then $u \equiv 0$. In particular, an $\mathcal{N}$-parabolic manifold is $\mathcal{D}$-parabolic.

In the theory of minimal surfaces in the Euclidean space, $\mathcal{D}$-parabolicity is not the only global property of surfaces with boundary that has been studied. Another property of interest is the quadratic area growth with respect to the extrinsic distance (see [6, 26, 24] for more details and applications of this property). To be more precise, we say that a surface $\Sigma \subset \mathbb{R}^3$ has *quadratic area growth* if, for some $C > 0$ and $A > 0$, one has

$$\mathrm{vol}(\Sigma \cap \{\sqrt{x_1^2 + x_2^2 + x_3^2} < R\}) \leq CR^2,$$

for all $R > A$.

The notions of $\mathcal{D}$-parabolicity and quadratic area growth seem to be, in general, unrelated concepts. For this reason, this global properties have been studied separetely in the theory of minimal surfaces in $\mathbb{R}^3$. However, according to Proposition 3.4, the volume condition

$$\int^{+\infty} \frac{R}{\mathrm{vol}(B_R(o))} = +\infty,$$

is sufficient to guarantee that a complete Riemannian manifold Σ is $\mathcal{D}$-parabolic. Hence, all the results obtained in this setting under geometric conditions on the ambient space and exploiting $\mathcal{D}$-parabolicity can be obtained imposing a volume growth condition on the surface instead. Moreover, since the volume of intrinsic balls is dominated by that of extrinsic balls with the same radius, we conclude also that any complete (e.g. properly immersed) surface in the Euclidean space with quadratic area growth is $\mathcal{D}$-parabolic.

To give an example of how this circle of ideas applies we note that it was conjectured by W. Meeks that any complete (or properly embedded) minimal graph over a proper subdomain of the plane is $\mathcal{D}$-parabolic. In [25], using refined stochastic methods, R. Neel gave a positive answer to this conjecture. Actually, he was able to prove that for a complete, embedded minimal surface with boundary whose Gauss image is eventually contained in a hyperbolic domain of the sphere, the Brownian motion strikes the boundary almost surely in finite time. However, apparently, no proofs based on analytic techniques of this fact has appeared yet in literature.

Nevertheless, according to Remark 2.2, minimal graphs Σ in $\mathbb{R}^{n+1}$ supported on a domain $\Omega \subset \mathbb{R}^n$ which either vanish (or more generally have constant boundary values) on the boundary, or such that $\langle \mathcal{N}, \mathcal{N}_0 \rangle$ has constant sign along the boundary, have the following volume growth property

$$\mathrm{vol}\left(B_R^\Sigma(o)\right) \le CR^n,$$

where $B_R^\Sigma(o)$ denotes the geodesic ball in M of radius R centered at a reference point $o \in \mathrm{int}\Sigma$. In particular, for a complete minimal graph Σ in the Euclidean 3-space which satisfies one of the conditions listed above,

$$\mathrm{vol}\left(B_R^\Sigma(o)\right) \le CR^2.$$

In view of Proposition 3.4, we have then proved the following theorem, that recovers the result by Neel in the two special cases considered above.

Theorem 3.5. *Any complete minimal graph Σ in $\mathbb{R}^3$ defined on a domain of the plane which either has constant boundary values or is such that $\langle \mathcal{N}, \mathcal{N}_o \rangle$ has constant sign along $\partial\Sigma$ is $\mathcal{D}$-parabolic.*

4. The L^2-Stokes theorem & slice-type results

In this section we prove the global divergence theorem stated in the Introduction as Theorem 0.15. We also provide a somewhat weaker form of this result which involves differential inequalities of the type $\mathrm{div}\, X \ge f$; see Proposition 4.5 below. This latter, together with the Ahlfors maximum principle, is then applied to prove slice-type results for hypersurfaces in product spaces and for graphs; see Theorems 0.16 and 0.17 in the Introduction. Actually, the graph-version of this result also requires a Liouville-type theorem for the mean curvature operator on manifolds with boundary, under volume growth conditions. This is modeled on [30].

4.1. Global divergence theorems.

Recall that, for a given smooth, compactly supported vector field X on an oriented Riemannian manifold $(M, \langle\, , \rangle)$ with boundary $\partial M \neq \emptyset$, the ordinary Stokes theorem asserts that

$$\int_M \mathrm{div}\, X = \int_{\partial M} \langle X, \nu \rangle, \tag{16}$$

where ν is the exterior unit normal to ∂M. In particular, this holds for every smooth vector field if M is compact. The result still holds if we relax the regularity conditions on X up to interpret its divergence in the sense of distributions. To be precise, we introduce the following definition.

Definition 4.1. *Let X be a vector field on M satisfying $|X| \in L^1_{loc}(M)$ and $\langle X, \nu \rangle \in L^1_{loc}(\partial M)$. The distributional divergence of X is defined by*

$$(\mathrm{div}\, X, \varphi) = -\int_M \langle X, \nabla\varphi \rangle + \int_{\partial M} \varphi \langle X, \nu \rangle, \tag{17}$$

for every $\varphi \in C_c^\infty(M)$.

Remark 4.2. The above definition extends trivially to $\varphi \in \mathrm{Lip}_c(M)$. Actually, more is true. Recall that, given a domain $D \subseteq M$, $W_0^{1,2}(D)$ denotes the closure of $C_c^\infty(D)$ in $W_0^{1,2}(D)$. Then, by a density argument, the previous definition extends to every $\varphi \in C_c^0(M) \cap W_0^{1,2}(M)$. Indeed, let φ be such a function. Then, we find an approximating sequence $\varphi_n \in C_c^\infty(M)$ such that $\varphi_n \to \varphi$ in $W^{1,2}(\mathrm{int} M)$, as $n \to +\infty$. Since $\mathrm{supp}(\varphi)$ is compact, we can assume that there exists a domain $\Omega \subset\subset M$ such that $\mathrm{supp}(\varphi_n) \subset \Omega$, for every n. Moreover, a subsequence (still denoted by φ_n) converges pointwise a.e. to φ. Let $c = \max_M |\varphi| + 1$ and define $\phi_n = f \circ \varphi_n \in \mathrm{Lip}_c(M)$ where

$$f(t) = \begin{cases} c, & t \geq c \\ t, & -c < t < c \\ -c, & t \leq -c. \end{cases}$$

Note that $\{\phi_n\}$ is an equibounded sequence, $\mathrm{supp}(\phi_n) \subset \Omega$ and, furthermore, $\phi_n \to f \circ \varphi = \varphi$ in $W^{1,2}(\mathrm{int} M)$ and pointwise a.e. in M. Therefore, evaluating (17) along ϕ_n, taking limits as $n \to +\infty$ and using the dominated convergence theorem completes the proof.

Now, suppose also that $\mathrm{div}\, X \in L^1_{loc}(M)$. Then we can write

$$(\mathrm{div}\, X, \varphi) = \int_M \varphi \,\mathrm{div}\, X$$

and, therefore, from (17) we get

$$\int_M \varphi \,\mathrm{div}\, X = -\int_M \langle X, \nabla\varphi \rangle + \int_{\partial M} \varphi \langle X, \nu \rangle.$$

In particular, if X is compactly supported , by choosing $\varphi = 1$ on the support of X, we recover the Stokes formula (16) for every compactly supported vector field X satisfying $X \in L^1_{loc}(M)$, $\mathrm{div}\, X \in L^1_{loc}(M)$ and $\langle X, \nu \rangle \in L^1_{loc}(\partial M)$.

Note that, by similar reasonings, if the vector field $|X| \in L^1_{loc}(M)$ has a weak divergence $\mathrm{div}\, X \in L^1_{loc}(M)$ and $\langle X, \nu \rangle \in L^1_{loc}(\partial M)$, then, for every $\rho \in C_c^0(M) \cap W_0^{1,2}(M)$, we have that $\mathrm{div}(\rho X) \in L^1_{loc}(M)$. Moreover, as in the smooth case,

$$\int_M \mathrm{div}(\rho X) = \int_M \langle \nabla\rho, X \rangle + \int_M \rho \,\mathrm{div}\, X.$$

To see this, we take $\varphi \in C_c^\infty(M)$ and, using (17) in the form of Remark 4.2, we compute

$$\begin{aligned}(\operatorname{div}(\rho X), \varphi) &= -\int_M \langle \rho X, \nabla\varphi\rangle + \int_{\partial M} \rho\varphi \langle X, \nu\rangle \\ &= -\int_M \langle X, \nabla(\rho\varphi)\rangle + \int_{\partial M} \rho\varphi\langle X,\nu\rangle + \int_M \varphi\langle X, \nabla\rho\rangle \\ &= (\operatorname{div} X, \rho\varphi) + \int_M \varphi \langle X, \nabla\rho\rangle \\ &= \int_M (\rho \operatorname{div} X + \langle X, \nabla\rho\rangle)\,\varphi \\ &= (\rho\operatorname{div} X + \langle X, \nabla\rho\rangle, \varphi).\end{aligned}$$

Whence, we conclude that

$$\operatorname{div}(\rho X) = \rho\operatorname{div} X + \langle X, \nabla\rho\rangle \in L^1_{loc}(M)$$

as desired.

All these facts will be tacitly employed several times in the rest of the Section.

If M is not compact, we can still prove a global version of Stokes theorem for vector fields with prescribed asymptotic behavior at infinity. This is the content of Theorem 0.15.

Proof (of Theorem 0.15). Suppose M is parabolic. According to Theorem 1.5 (ii) there exist as exhaustion $\{\Omega_n\}$ and an increasing sequence of functions $\varphi_n \in C_c(M) \cap W^{1,2}(\mathrm{int}M)$ supported in Ω_n such that $0 \le \varphi_n \le 1$ and

$$\varphi_n \to 1 \text{ locally uniformly on } M \text{ and } \int_M |\nabla\varphi_n|^2 \to 0.$$

Consider now any vector field X satisfying (5). Since $\varphi_n X$ is compactly supported, applying the usual (weak) divergence theorem we get

$$\int_M \operatorname{div}(\varphi_n X) = \int_{\Omega_n} \operatorname{div}(\varphi_n X) = \int_{\partial_1\Omega_n} \varphi_n \langle X, \nu\rangle. \tag{18}$$

On the other hand

$$\int_M \operatorname{div}(\varphi_n X) = \int_M \langle \nabla\varphi_n, X\rangle + \int_M \varphi_n \operatorname{div} X,$$

where

$$\left|\int_M \langle\nabla\varphi_n, X\rangle\right| \le \left(\int_M |\nabla\varphi_n|^2\right)^{\frac12} \left(\int_M |X|^2\right)^{\frac12} \to 0$$

as $n \to +\infty$. Moreover

$$\int_M \varphi_n \operatorname{div} X = \int_M \varphi_n (\operatorname{div} X)_+ - \int_M \varphi_n(\operatorname{div} X)_-$$

and

$$\int_M \varphi_n (\operatorname{div} X)_+ \le \int_M \varphi_n (\operatorname{div} X)_- + \int_{\partial_1 \Omega_n} \varphi_n \langle X, \nu \rangle - \int_M \langle \nabla \varphi_n, X \rangle .$$

Using the monotone and dominated convergence theorems and the fact that $0 \le \varphi_n \le 1$, we obtain

$$\int_M (\operatorname{div} X)_+ \le \int_M (\operatorname{div} X)_- + \int_{\partial_1 M} \langle X, \nu \rangle < +\infty.$$

Hence $\operatorname{div} X \in L^1(M)$ and taking limits on both sides of (18) completes the first part of the proof.

Conversely, assume that M is not parabolic so that M possesses a smooth, finite, positive Green kernel, [10, 12]. We shall show that the global Stokes theorem fails. To this end, choose an exhaustion $\{\Omega_n\}$ of M by smooth and relatively compact domains. Then, the Neumann Green kernel $G(x,y)$ of M is obtained as the limit of the Green functions $G_n(x,y)$ of Ω_n which satisfy

$$\begin{cases} \Delta G_n(x,y) = -\delta_x(y) & \text{on } \operatorname{int}\Omega_n \\ \dfrac{\partial G_n}{\partial \nu} = 0 & \text{on } \partial_1 \Omega_n \\ G_n = 0 & \text{on } \partial_0 \Omega_n. \end{cases}$$

Let $0 \le f \not\equiv 0$ be a smooth function compactly supported in $\operatorname{int}M$. For each n define

$$u_n(x) = \int_{\Omega_n} G_n(x,y) f(y)\, dy.$$

Then, each u_n is a positive, classical solution of the boundary value problem

$$\begin{cases} \Delta u_n = -f & \text{on } \operatorname{int}\Omega_n \\ \dfrac{\partial u_n}{\partial \nu} = 0 & \text{on } \partial_1 \Omega_n \\ u_n = 0 & \text{on } \partial_0 \Omega_n. \end{cases}$$

By the maximum principle and the boundary point lemma, the sequence is monotonically increasing and converges to a solution u of

$$\begin{cases} \Delta u = -f & \text{on } \operatorname{int}M \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial M. \end{cases}$$

Also, using Fatou Lemma,

$$\int_M |\nabla u_n|^2 \ge \int_M |\nabla u|^2 .$$

Now consider the vector field

$$X = \nabla u.$$

Clearly X satisfies all the conditions in (5). On the other hand, we have

$$\int_M \operatorname{div} X = -\int_M f \neq 0$$

and

$$\int_{\partial M} \langle X, \nu \rangle = \int_{\partial M} \frac{\partial u}{\partial \nu} = 0,$$

proving that the global Stokes theorem fails to hold. □

Using Definition 4.1 of weak divergence one could introduce the notion of weak solution of a differential inequality like $\operatorname{div} X \geq f$. We stress that $\operatorname{div} X$ is not required to be a function.

Definition 4.3. *Let $|X| \in L^1_{loc}(M)$ be a vector field satisfying $\langle X, \nu \rangle \in L^1_{loc}(\partial M)$ and let $f \in L^1_{loc}(M)$. We say that $\operatorname{div} X \geq f$ in the distributional sense on $\operatorname{int} M$ if*

$$(\operatorname{div} X, \varphi) \geq \int_M f\varphi,$$

for every $0 \leq \varphi \in C^\infty_c(M)$. Actually, according to Remark 4.2, the definition extends to every $0 \leq \varphi \in C^0_c(M) \cap W^{1,2}(\operatorname{int} M)$.

In the special case where $f = 0$ and $X = \nabla u$ for some $u \in W^{1,2}_{loc}(M)$ satisfying $\partial u/\partial \nu \in L^1_{loc}(\partial M)$, we obtain the corresponding notion of weak solution of $\Delta u \geq 0$ on $\operatorname{int} M$.

Although elementary, it is important to realize that, as in the smooth setting, the above definition is compatible with that of weak Neumann subsolution given in the Introduction.

Lemma 4.4. *Let $u \in W^{1,2}_{loc}(M)$ satisfy $\partial u/\partial \nu \in L^1_{loc}(\partial M)$. Then u is a weak Neumann subsolution of the Laplace equation provided u satisfies*

$$\begin{cases} \Delta u \geq 0 & \text{on } \operatorname{int} M \\ \dfrac{\partial u}{\partial \nu} \leq 0 & \text{on } \partial M, \end{cases}$$

where the differential inequality is interpreted according to Definition 4.3.

Proof. Straightforward from the equation

$$(\Delta u, \varphi) \overset{\text{def}}{=} -\int_M \langle \nabla u, \nabla \varphi \rangle + \int_{\partial M} \frac{\partial u}{\partial \nu}\varphi,$$

with $0 \leq \varphi \in C^\infty_c(M)$. □

Reasoning as in the proof of Theorem 0.15, we can now prove the following result which extends to manifolds with boundary a result in [15].

Proposition 4.5. *Let $(M, \langle\, , \rangle)$ be an m-dimensional, parabolic manifold with smooth boundary ∂M. Let X be a vector field on M satisfying:*

$$\text{(a) } |X| \in L^2(M)\,; \text{ (b) } 0 \geq \langle X, \nu \rangle \in L^1_{loc}(\partial M)\,.$$

Assume that $\operatorname{div} X \geq f$ for some $f \in L^1(M)$ in the sense of distributions. Then

$$\int_M f \leq \int_{\partial M} \langle X, \nu \rangle.$$

The same conclusion holds if $0 \le f \in L^1_{loc}(M)$ *and yields*

$$f \equiv 0.$$

Moreover, if $\operatorname{div} X \ge 0$ *in the distributional sense, then*

$$\int_M \langle X, \nabla\alpha\rangle \le \int_{\partial M} \alpha \langle X, \nu\rangle$$

for every $0 \le \alpha \in C^\infty_c(M)$.

Proof. Choose a smooth, relatively compact exhaustion $\{\Omega_n\}$ of M and denote by φ_n the equilibrium potential of the capacitor $(\overline{\Omega}_0, \Omega_n)$. Extend φ_n to be identically 1 on Ω_0 and identically 0 on $M \backslash \Omega_n$. Then, $\varphi_n \nearrow 1$ pointwise on M and, since M is parabolic, $\|\nabla\varphi_n\|_{L^2} \to 0$ as $n \to +\infty$. Since, by assumption,

$$\begin{aligned}\int_M \varphi_n f &\le (\operatorname{div} X, \varphi_n)\\ &= -\int_M \langle X, \nabla\varphi_n\rangle + \int_{\partial M} \varphi_n \langle X, \nu\rangle\\ &\le \left(\int_M |X|^2\right)^{\frac{1}{2}} \left(\int_M |\nabla\varphi_n|^2\right)^{\frac{1}{2}} + \int_{\partial M} \varphi_n\langle X, \nu\rangle,\end{aligned}$$

the first part of the statement follows by taking the limit as $n \to +\infty$ and applying the monotone convergence theorem to boundary integral, and either the monotone convergence or the dominated convergence theorem to the left hand side, depending on whether $0 \le f \in L^1_{loc}(M)$ or if $f \in L^1(M)$.

For what concerns the second part, consider the test function $\eta = \varphi_n\alpha$. Then,

$$\begin{aligned}0 &\le (\operatorname{div} X, \alpha\varphi_n)\\ &= -\int_M \alpha\langle X, \nabla\varphi_n\rangle - \int_M \varphi_n\langle X, \nabla\alpha\rangle + \int_{\partial M} \alpha\varphi_n\langle X, \nu\rangle\\ &\le \sup_M |\alpha| \left(\int_M |X|^2\right)^{\frac{1}{2}} \left(\int_M |\nabla\varphi_n|^2\right)^{\frac{1}{2}} - \int_M \varphi_n\langle X, \nabla\alpha\rangle + \int_{\partial M} \alpha\varphi_n\langle X, \nu\rangle.\end{aligned}$$

and the conclusion follows as above computing the lim sup as $n \to +\infty$. □

4.2. **Slice-type theorems for hypersurfaces in a half-space.** This Section is devoted to the proofs of Theorems 0.16 and 0.17 stated in the Introduction. The first one of these results involves a complete hypersurface Σ contained in the half-space $N \times [0 + \infty)$ of the ambient product space $N \times \mathbb{R}$. It is assumed that the boundary $\partial\Sigma \neq \emptyset$ lies in the slice $N \times \{0\}$ and that Σ has non-positive mean curvature $H \le 0$ with respect to the "downward"Gauss map. The result states that, under a quadratic area growth assumption on Σ and regardless of the geometry of N, the portion of the hypersurface Σ in any upper-halfspace of $N \times \mathbb{R}$ must have infinite volume unless

Σ is contained in the totally geodesic slice $N \times \{0\}$. The second result provides a graphical version of this theorem when $\Sigma = \Gamma_u(M)$, where $(M, \langle\, ,\rangle)$ is a complete oriented manifold with boundary. If M satisfies a quadratic volume growth assumption, then each superlevel set $M_t = \{u \geq t > 0\} \subseteq M$ has infinite volume unless Σ is contained in the totally geodesic slice $M \times \{0\}$. Note that M_t is the orthogonal projection of $\Sigma \cap [t, +\infty)$ on the slice $M \times \{0\}$.

Let us begin with the

Proof (of Theorem 0.16). We start off remarking that Σ is parabolic because of the volume growth assumption. Suppose that Σ is not contained in the slice $N \times \{0\}$. If the height function h on Σ is bounded from above (for the precise definition of h see the proof of Theorem 0.11 in Subsection 2.2) the parabolicity of Σ in the form of the Ahlfors maximum principle implies that

$$h \leq \sup_{\Sigma} h = \sup_{\partial\Sigma} h = 0.$$

The conclusion is then immediate because, by assumption, Σ is contained in the half-space $N \times [0, +\infty)$. Suppose now that $\sup_{\Sigma} h = +\infty$, so that $\Sigma \cap (N \times \{t\}) \neq \emptyset$ for arbitrary $t > 0$. Letting

$$\Sigma_t = \Sigma \cap (N \times [t, +\infty)) = \{p \in \Sigma : h(p) \geq t\},$$

and since $\mathrm{vol}(\Sigma_t) \geq \mathrm{vol}(\Sigma_s)$, for every $s \geq t$, we can assume that $\mathrm{vol}(\Sigma_t) < +\infty$ for every $t >> 1$. Moreover, by Sard theorem we can suppose that t is a regular value of $h|_{\mathrm{int}\Sigma}$. In particular, Σ_t is a smooth complete hypersurface with boundary $\partial\Sigma_t = \{p \in \Sigma : h(p) = t\}$ and exterior unit normal $\nu_t = -\nabla h / |\nabla h|$. Clearly, Σ_t is parabolic because it has finite volume. According to (15), h is a subharmonic function on Σ_t and satisfies $|\nabla h| \leq 1$. In particular, $|\nabla h| \in L^2(\Sigma_t)$. For any $\varepsilon > 0$ define

$$h_\varepsilon = \max\{h, t + \varepsilon\}.$$

Then h_ε is again subharmonic on Σ_t, it has finite Dirichet energy $|\nabla h_\varepsilon| \in L^2(\Sigma_t)$ and, furthermore, $\partial h_\varepsilon / \partial \nu = 0$ on $\partial\Sigma_t$. Therefore, we can apply Proposition 4.5 and deduce that h_ε has to be harmonic on Σ_t. Actually, since h_ε is bounded from below on the parabolic manifold Σ_t it follows that h_ε is constant on every connected component of Σ_t. Whence, on noting that $h_\varepsilon = t + \varepsilon$ on $\partial\Sigma_t$ we obtain that $t \leq h \leq t + \varepsilon$ on Σ_t. Since this holds for every $\varepsilon > 0$ we conclude that $h \equiv t$ on Σ_t, contradicting the assumption of h being unbounded. □

The proof of Theorem 0.17 is completely similar but requires some preparation. The next Liouville-type result for the mean curvature operator is adapted from [30]; see also [5, 2]. We provide a detailed proof for the sake of completeness.

Theorem 4.6. *Let (M,g) be a complete Riemannian manifold with boundary $\partial M \neq \emptyset$. If, for some reference point $o \in \mathrm{int}M$,*

$$\int^{+\infty} \frac{1}{\mathrm{Area}\left(\partial_0 B_R\left(o\right)\right)} = +\infty, \tag{19}$$

then the following holds. Let $u \in C^1\left(M\right)$ be a weak Neumann solution of the problem

$$\begin{cases} \mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+\left|\nabla u\right|^2}}\right) \geq 0 & \text{on } \mathrm{int}M \\ \dfrac{\partial u}{\partial \nu} \leq 0 & \text{on } \partial M \\ \sup_M u < +\infty. \end{cases} \tag{20}$$

Then $u \equiv \mathrm{const}$.

Remark 4.7. As already pointed out for the Laplace-Beltrami operator, being a weak Neumann solution of $\mathrm{div}((1+|\nabla u|^2)^{-1/2}\nabla u)) \geq 0$ means that

$$-\int_{\mathrm{int}M} \left\langle \frac{\nabla u}{\sqrt{1+\left|\nabla u\right|^2}}, \nabla\varphi \right\rangle \geq 0, \tag{21}$$

for every $0 \leq \varphi \in C_c^\infty\left(M\right)$. As usual, by density, inequality (21) can be extended to compactly supported test functions $0 \leq \varphi \in W^{1,2}(\mathrm{int}M)$. Actually, it is obvious that the same definition extends to any elliptic operator of the form $L_\Phi\left(u\right) = \mathrm{div}(\Phi(|\nabla u|)\nabla u)$, where $\Phi\left(t\right)$ is subjected to certain structural conditions. Moreover, under the assumption

$$\left|\nabla u\right| \in L^1_{loc}\left(\partial M\right),$$

this definition is also coherent with the notion of weak divergence. Namely u satisfies (21) provided $(\mathrm{div}\, X, \varphi) \geq 0$ and $\partial u/\partial\nu \leq 0$, where we have set $X = (1+|\nabla u|^2)^{-1/2}\nabla u$. This follows immediately from the equation

$$(\mathrm{div}\, X, \varphi) \overset{\mathrm{def}}{=} -\int_{\mathrm{int}M} \left\langle \frac{\nabla u}{\sqrt{1+\left|\nabla u\right|^2}}, \nabla\varphi \right\rangle + \int_{\partial M} \frac{\varphi}{\sqrt{1+\left|\nabla u\right|^2}} \frac{\partial u}{\partial \nu}.$$

Remark 4.8. If we take $\Phi\left(t\right) = 1$ in the argument below we recover Theorem 0.7 by Grigor'yan, in the form of a Liouville result for $C^1(M)$ subsolutions of the Laplace equation.

Proof. Let u be as in the statement of the theorem and assume, by contradiction, that u is non-constant on the ball $B_{R_0}(o)$, for some $R_0 > 0$. Without loss of generality we can suppose that $u \leq 0$ on M. Define

$$\Phi\left(t\right) = \frac{1}{\sqrt{1+t^2}}.$$

Now, having fixed $R > R_0$ and $\varepsilon > 0$, we choose $\rho = \rho_{\varepsilon,R}$ as follows:

$$\rho(x) = \begin{cases} 1 & \text{on} \quad B_R(o) \\ \frac{R+\varepsilon-r(x)}{\varepsilon} & \text{on} \quad B_{R+\varepsilon}(o)\backslash B_R(o) \\ 0 & \text{elsewhere.} \end{cases}$$

Inserting the test function $\varphi = \rho \mathrm{e}^u$ into (21) and rearranging we get

$$\begin{aligned} 0 &\le -\int_{\mathrm{int}M} \langle \Phi(|\nabla u|)\nabla u, \nabla(\rho \mathrm{e}^u)\rangle \\ &= -\int_{\mathrm{int}M} \mathrm{e}^u \Phi(|\nabla u|) \langle \nabla u, \nabla \rho\rangle - \int_{\mathrm{int}M} \rho \mathrm{e}^u \Phi(|\nabla u|)\, |\nabla u|^2 . \end{aligned}$$

Then,

$$\varepsilon^{-1} \int_{(B_{R+\varepsilon}(o)\backslash B_R(o))\cap \mathrm{int}M} \mathrm{e}^u \Phi(|\nabla u|)\langle \nabla u, \nabla r\rangle \ge \int_{B_R(o)\cap \mathrm{int}M} \mathrm{e}^u \Phi(|\nabla u|)\, |\nabla u|^2 .$$

Using the co-area formula and letting $\varepsilon \to 0$ we get, for a.e. $R > R_0$,

$$\int_{\partial_0 B_R(o)} \mathrm{e}^u \Phi(|\nabla u|)\langle \nabla u, \nabla r\rangle \ge \int_{B_R(o)\cap \mathrm{intM}} \mathrm{e}^u \Phi(|\nabla u|)|\nabla u|^2.$$

On the other hand, using the Cauchy-Schwartz and Hölder inequalities, we obtain

$$\begin{aligned} \int_{\partial_0 B_R(o)} \mathrm{e}^u \Phi(|\nabla u|)\langle \nabla u, \nabla r\rangle &\le \int_{\partial_0 B_R(o)} \mathrm{e}^u \Phi(|\nabla u|)\, |\nabla u| \\ &\le \left(\int_{\partial_0 B_R(o)} \mathrm{e}^u \Phi(|\nabla u|)\right)^{\frac{1}{2}} \left(\int_{\partial_0 B_R(o)} \mathrm{e}^u \Phi(|\nabla u|)\, |\nabla u|^2\right)^{\frac{1}{2}} \\ &\le \mathrm{Area}(\partial_0 B_R(o))^{\frac{1}{2}} \left(\int_{\partial_0 B_R(o)} \mathrm{e}^u \Phi(|\nabla u|)\, |\nabla u|^2\right)^{\frac{1}{2}} . \end{aligned}$$

Now, set

$$H(R) = \int_{B_R(o)\cap \mathrm{intM}} \mathrm{e}^u \Phi(|\nabla u|)\, |\nabla u|^2 ,$$

Since u is assumed to be non-constant on $B_{R_0}(o)$ then $H(R) \neq 0$ for every $R \ge R_0$. Then, by the co-area formula and the previous inequalities,

$$\frac{H'(R)}{H(R)^2} \ge \frac{1}{\mathrm{Area}(\partial_0 B_R(o))}.$$

Integrating this latter on $[R_0, R]$ and letting $R \to +\infty$ we conclude

$$H(R_0) \le \frac{1}{\int_{R_0}^{+\infty} \mathrm{Area}(\partial_0 B_R(o))^{-1}} = 0,$$

proving that

$$\int_{B_{R_0}(o)\cap \mathrm{intM}} \mathrm{e}^u \Phi(|\nabla u|)\, |\nabla u|^2 = 0.$$

Therefore, u must be constant on $B_{R_0}(o)$, leading to a contradiction. □

We are now ready to prove the slice theorem for graphs.

Proof (of Theorem 0.17). Let $\Sigma = \Gamma_u(M)$, with $u \in C^\infty(M)$ and, for every $s \in \mathbb{R}$, define

$$M_s := \{x \in M : u(x) \geq s\}.$$

By the assumption on $\partial\Sigma = \Gamma_u(\partial M)$, there exists $t > 0$ such that, for every $s \geq t$, $M_s \subset\subset \mathrm{int}M$ and $\mathrm{vol}(M_s) < +\infty$. Assume that $M_s \neq \emptyset$ for every $s \geq t$, for otherwise, as in Theorem 0.16, the proof is easier. Then, by Sard Theorem, we can choose $t < c < \sup_M u \leq +\infty$ such that c is a regular value of $u|_{\mathrm{int}M}$. Thus, the closed subset M_c is a complete manifold with boundary $\partial M_c \neq \emptyset$ and exterior unit normal $\nu_c = -\nabla u/|\nabla u|$. In particular, as a complete manifold with finite volume, M_c is parabolic. Since the smooth function u satisfies

$$\mathrm{div}\left(\frac{\nabla_M u}{\sqrt{1+|\nabla_M u|^2}}\right) = -mH \geq 0, \text{ on } \mathrm{int}M_c$$

then, having fixed any $\varepsilon > 0$, the same differential inequality holds, in a weak sense, for

$$u_\varepsilon = \max\{u, c+\varepsilon\};$$

see Lemma A.4. Note also that $\partial u_\varepsilon/\partial\nu = 0$ on ∂M_c. Summarizing, the vector field

$$X_\varepsilon = \frac{\nabla_M u_\varepsilon}{\sqrt{1+|\nabla_M u_\varepsilon|^2}}$$

satisfies

$$\begin{cases} \mathrm{div}_M X_\varepsilon \geq 0 & \text{on } \mathrm{int}M_c \\ 1 \geq |X_\varepsilon| \in L^2(M_c) & \\ 0 = \langle X_\varepsilon, \nu_c\rangle. & \end{cases}$$

By applying Proposition 4.5 we deduce that $\mathrm{div}_M X = 0$ on M_c, i.e., $\Sigma_c = \Gamma_u(M_c)$ is a minimal graph. Actually, since $\mathrm{vol}(M_c) < +\infty$, by Theorem 4.6 we get that u_ε must be constant on every connected component of M_c. Since $u_\varepsilon = c+\varepsilon$ on ∂M_c it follows that $c \leq u \leq c+\varepsilon$ on M_c. Whence, using the fact that $\varepsilon > 0$ was chosen arbitrarily, we conclude that $u \equiv c$ on M_c. This contradicts the fact that $M_s \neq \emptyset$.

Since u is constant on M_c we have that $\sup_M u < +\infty$. We now distinguish three cases.

(a) Suppose that $\partial\Sigma \subset M \times \{0\}$ and $\Sigma \subset [0, +\infty)$. This means that $u \geq 0$ with $u = 0$ on ∂M. In this case the conclusion $u \equiv 0$ follows exactly as in proof of Theorem 0.16.

(b) Suppose that Σ is real analytic, i.e., it is described by a real analytic function u. Since u is constant on the interior of M_c we must conclude that u is constant everywhere.

(c) Suppose that $\cos\widehat{\mathcal{N}_0\mathcal{N}} \leq 0$ on $\partial\Sigma = \Gamma_u(\partial M)$. This means that $\partial u/\partial\nu \leq 0$ on ∂M. The desired conclusion follows by a direct application of Theorem 4.6. □

The following corollary is a straightforward consequence of the above proof.

Corollary 4.9. *Let $(M, \langle\, ,\rangle)$ be a complete manifold with boundary ∂M and assume that* $\mathrm{vol}(M) < +\infty$. *Let $\Sigma = \Gamma_u(M)$ be a graph with non-positive mean curvature $H(x)$ with respect to the downward Gauss map $\mathcal{N}$. Assume also that the angle θ between the Gauss map $\mathcal{N}$ of the graph Σ and the Gauss map $\mathcal{N}_0 = (-\nu, 0)$ of $\partial M \times \{t\} \hookrightarrow M \times \{t\}$ satisfies $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then Σ is a horizontal slice of $M \times \mathbb{R}$.*

Appendix A. Sobolev spaces and truncated subsolutions on manifolds with boundary

In this section we collect the Meyers-Serrin density results on manifolds with boundary and we recall how these imply that truncated weak subsolutions of the Neumann problem are again subsolutions.

Throughout this section, $(M, \langle\, ,\rangle)$ will always denote a (non-necessarily complete) m-dimensional manifold with smooth boundary $\partial M \neq \emptyset$. The corresponding Sobolev spaces are those defined in the Introduction of the paper.

We begin with the following

Theorem A.1 (Meyers-Serrin, I)**.** *Let $D \subseteq M$ be a domain. Then*

$$W^{1,2}(\mathrm{int}(D)) = \overline{C^\infty(\mathrm{int}D) \cap W^{1,2}(\mathrm{int}D)}^{W^{1,2}}.$$

Proof. Indeed, $N = \mathrm{int}D$ is a smooth manifold without boundary so we can select a countable, locally finite, smooth atlas $\{V_k \to \mathbb{R}^m\}$ with $\overline{V}_k$ compact. Using the shrinking Lemma, we find a locally finite, open covering $\{U_k\}$ such that $\overline{U}_k \subset V_k$. We define a partition of unity $\{\chi_k\}$ subordinated to the covering $\{U_k\}$ and decompose $u \in W^{1,2}(N)$ as $u = \sum_k u_k$ with $u_k = u \cdot \chi_k$.

Now, having fixed $\varepsilon > 0$, and using mollifiers, we find $f_k \in C^\infty_c(U_k)$ such that

$$\|u_k - f_k\|_{W^{1,2}} \leq \frac{\varepsilon}{2^k}.$$

Thus, the locally finite sum $f = \sum_k f_k$ is smooth on N and gives an ε-approximation of u in the space $W^{1,2}(\mathrm{int}M)$. □

Next we show that the smooth approximation can in fact be achieved using functions which are smooth up to the boundary $\partial_1 D$. Moreover, compactly supported functions can be approximated using approximations with support in an arbitrarily small negihborhood of the support of the function.

Theorem A.2 (Meyers-Serrin, II)**.** *Let $D \subseteq M$ be a domain. Then*

$$W^{1,2}(\mathrm{int}(D)) = \overline{C^\infty(D) \cap W^{1,2}(\mathrm{int}D)}^{W^{1,2}}.$$

Moreover, if $f \in W^{1,2}(\mathrm{int}D)$ *has compact support contained* D*, there exists a sequence* $\{f_n\} \in C_c^\infty(M)$ *such that*

$$\|f - f_n\|_{W^{1,2}(\mathrm{int}D)} \to 0,$$

and it can be arranged that $\mathrm{supp}f_n$ *is contained in an* ε*-neighborhood of* $\mathrm{supp}f$*.*

Proof. The proof goes exactly as in the Myers-Serrin density result up to recalling that, by the definition of smoothness for objects on a manifold with boundary, at each fixed point $p \in \partial_1 D$ the Riemannian metric of M has a small coordinate neighborhood extension past its boundary. Therefore, we can select a smooth atlas $\{V_k \to \mathbb{H}^m_+\}$ whose boundary charts enjoy this extension property, consider the corresponding (shrinked) relatively compact covering $\{U_k\}$, and extend each function u_k (in local coordinates) to a $W^{1,2}$-function $\tilde{u}_k$ on an open set of $\mathbb{R}^m$, before approximating it with some $f_k \in C_c^\infty(U_k)$. Thus $f = \sum_k f_k$ is smooth on D and $W^{1,2}$-approximate u on $\mathrm{int}D$. The second assertion follows from the fact that, by compactness, only finitely many open sets U_k intersect $\mathrm{supp}f$. □

By using the above result and the standard uniformly Lipschitz sequence of cut-off functions $\{\chi_n\}$defined by

$$\chi_n(x) = \begin{cases} 1 & \text{if } r(x) \le n \\ n+1-r(x) & \text{if } n \le r(x) \le n+1 \\ 0 & \text{if } r(x) \ge n+1, \end{cases}$$

where $r(x)$ is the distance function from a fixed point $o \in \mathrm{int}M$, it is easy to show that if M is complete then

$$W^{1,2}(\mathrm{int}M) = \overline{\mathrm{Lip}_c(M)}^{W^{1,2}}.$$

Indeed, let $u \in W^{1,2}(\mathrm{int}M)$. Without loss of generality, by the first half of Theorem A.2, we can assume that $u \in C^\infty(M)$. Theorefore, $u_n = \chi_n u \in \mathrm{Lip}_c(M) \subset W^{1,2}(\mathrm{int}M)$ and its distributional gradient satisfies the product rule $\nabla u_n = u\nabla\chi_n + \chi_n \nabla u$ in $\mathrm{int}M$. Since $u \in W^{1,2}(\mathrm{int}M)$, on noting that $\|\nabla\chi_n\|_\infty \to 0$ and $\chi_n \to 1$ as $n \to \infty$, we can use dominated convergence and conclude that $\|u - u_n\|_{W^{1,2}} \to 0$, as $n \to \infty$.

Combining this with Theorem A.2 we deduce

Corollary A.3. *Let* M *be a complete Riemannian manifold. Then*

$$W^{1,2}(\mathrm{int}M) = \overline{C_c^\infty(M)}^{W^{1,2}}.$$

Proof. Let $u \in W^{1,2}(\mathrm{int}M)$. Having fixed $\varepsilon > 0$ we find $u_1 \in \mathrm{Lip}_c(M)$ such that $\|u - u_1\|_{W^{1,2}} < \varepsilon/2$. On the other hand, according to the second half of Theorem A.2, there exists $u_2 \in C_c^\infty(M)$ satisfying $\|u_1 - u_2\|_{W^{1,2}} < \varepsilon/2$. It then follows that u_2 is the desired ε-approximation of u in $W^{1,2}(\mathrm{int}M)$. □

The next lemma concerns with truncated subsolutions of the Neumann problem. We show that these are still subsolutions by adapting the well known proof for subsolutions of the Laplace equation on manifolds without boundary; see e.g. [28, Theorem C.1]. Indeed, such a proof relies almost exclusively on the Meyers-Serrin density results.

Lemma A.4. *Let $u \in C^0(M) \cap W^{1,2}_{\mathrm{loc}}(M)$ be a weak Neumann subsolution of the Laplace equation. Then, for every $a, b \in \mathbb{R}$, the function $v(x) = \max\{a + u(x), b\} \in C^0(M) \cap W^{1,2}_{\mathrm{loc}}(M)$ is a weak Neumann subsolution.*

Proof. We can limit ourselves to consider $v(x) = \max\{u(x), 0\} = u_+(x)$. We follow the standard arguments for the boundary-less case; see e.g. [28, Appendix C]. Clearly, $u_+ \in C^0(M) \cap W^{1,2}_{\mathrm{loc}}(M)$. Take any non-decreasing function $\lambda \in C^\infty(\mathbb{R})$ satisfying $0 \le \lambda(t) \le 1$, $\lambda(t) = 0$, $\forall t \le 0$ and $\lambda(t) = 1$, $\forall t \ge 1$. Define

$$\lambda_n(t) = \lambda(nt),$$

so that $\lambda_n(t) = 0$ for $t \le 0$, $\lambda_n(t) = 1$ if $t \ge 1/n$ and $0 \le \lambda_n'(t) \le Cn$ for every $t \in \mathbb{R}$ and for some universal constant $C > 0$. In particular, $\lambda_n(t) \to \mathbf{1}_{(0,+\infty)}(t)$ pointwise on $\mathbb{R}$. Having fixed $0 \le \rho \in C^\infty_c(M)$ let $\Omega \subset\subset M$ be a relatively compact domain with $\mathrm{supp}(\rho) \subset\subset \Omega$. Then, by Theorem A.2, there exists a sequence $\{u_n^+\} \in C^\infty_c(M)$ such that

$$u_n^+ \to u_+ \text{ in } W^{1,2}(\mathrm{int}\Omega).$$

Up to passing to a subsequence (still denoted by u_n) we can assume that

$$u_n^+ \to u_+ \text{ a.e. in } \Omega,$$

and

$$\|u_n^+ - u_+\|_{W^{1,2}(\mathrm{int}\Omega)} \le \frac{1}{n^2}. \tag{22}$$

Using the $W^{1,2}_0(M)$ test function

$$0 \le \phi = \lambda_n(u_n^+)\rho$$

in the definition of weak Neumann subsolution we get

$$\int_{\mathrm{int}M} \langle \nabla u, \nabla \rho \rangle \lambda_n(u_n^+) + \int_{\mathrm{int}M} \left\langle \nabla u, \nabla u_n^+ \right\rangle \lambda_n'(u_n^+)\rho \le 0.$$

Since

$$\left|\lambda_n\left(u_n^+\right) \langle \nabla u, \nabla \rho \rangle\right| \le |\nabla u|\,|\nabla \rho| \in L^1(\Omega),$$

by dominated convergence we have

$$\begin{aligned}\int_{\mathrm{int}M} \langle \nabla u, \nabla \rho \rangle \lambda_n(u_n^+) &\to \int_{\{u>0\}\cap\Omega} \langle \nabla u, \nabla \rho \rangle \\ &= \int_{\mathrm{int}M} \langle \nabla u_+, \nabla \rho \rangle.\end{aligned}$$

Thus, to conclude, it remains to prove that

$$\liminf_{n\to+\infty}\int_{\mathrm{int}M}\left\langle\nabla u,\nabla u_n^+\right\rangle\lambda_n'\left(u_n^+\right)\rho\geq 0. \tag{23}$$

To this end, by adding and subtracting ∇u_+, applying the Cauchy–Schwarz inequality, recalling that $\rho,\lambda_n'\geq 0$ and noting that $\langle\nabla u,\nabla u_+\rangle=|\nabla u_+|^2\cdot\mathbf{1}_{\{u>0\}}$, we compute

$$\begin{aligned}\int_{\mathrm{int}M}\left\langle\nabla u,\nabla u_n^+\right\rangle\lambda_n'\left(u_n^+\right)\rho &\geq -\int_{\mathrm{int}M}|\nabla u|\left|\nabla u_n^+-\nabla u_+\right|\lambda_n'\left(u_n^+\right)\rho\\ &\quad+\int_{\mathrm{int}M}\left\langle\nabla u,\nabla u_+\right\rangle\lambda_n'\left(u_n^+\right)\rho\\ &\geq -\int_{\mathrm{int}M}|\nabla u|\left|\nabla u_n^+-\nabla u_+\right|\lambda_n'\left(u_n^+\right)\rho.\end{aligned} \tag{24}$$

On the other hand, using the L^2 Cauchy–Schwarz inequality, $|\lambda_n'|\leq Cn$ and (22),

$$\begin{aligned}-\int_{\mathrm{int}M}|\nabla u|\left|\nabla u_n^+-\nabla u_+\right|\lambda_n'\left(u_n^+\right)\rho &\geq -Cn\int_{\mathrm{int}M}|\nabla u|\left|\nabla u_n^+-\nabla u^+\right|\rho\\ &\geq -Cn\left\|\nabla u\right\|_{L^2(\Omega)}\left\|\nabla u_n^+-\nabla u_+\right\|_{L^2(\Omega)}\\ &\geq -\frac{C}{n}\left\|\nabla u\right\|_{L^2(\Omega)}\to 0.\end{aligned}$$

Inserting this information into (24) shows the validity of (23) and completes the proof. □

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