ON THE GROWTH OF SUPERSOLUTIONS OF NONLINEAR PDE'S ON EXTERIOR DOMAINS

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Abstract. We obtain a comparison principle on annuli, with "catenoid-like" functions, for supersolutions of non-linear elliptic PDEs over exterior domains in a non-positively curved manifold with a pole. This result is applied to get an upper estimate on the growth of such supersolutions and, in particular, of exterior graphs of non-negative mean curvature.

Introduction

Let Ω be a domain in the smooth, m-dimensional Riemannian manifold (M, g). The graph of the smooth function $u : \Omega \to \mathbb{R}$ is the embedded hypersurface $\Sigma_u = \{(x, u(x)) : x \in \Omega\} \hookrightarrow (M \times \mathbb{R}, g + dt^2)$. Thus, u measures the height of the graph Σ_u from the 0-level slice $M \times \{0\}$. We orient Σ_u by the downward pointing unit normal defined by

$$\mathcal{N} = \frac{1}{\sqrt{1 + |\nabla u|^2}}(\nabla u, -1),$$

where the gradient, as well as the other differential operators appearing below, is computed with respect to the metric g of M. The mean curvature of the smooth graph Σ_u corresponding to the orientation $\mathcal{N}$ writes as

$$\mathcal{H}_u = -\frac{1}{m} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right).$$

The non-linear, divergence-form, differential operator $\operatorname{div}((1+|\nabla u|^2)^{-\frac{1}{2}}\nabla u)$ is usually referred to as the *mean curvature operator* on (M, g).

When $(M, g) = (\mathbb{R}^2, \mathrm{can})$ and $\Omega \subset \mathbb{R}^2$ is an unbounded domain, a classical result by P. Collin and R. Krust, [2], states that if $u, v \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ are

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distinct solutions of the problem

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) & \text{on } \Omega \\ u = v & \text{on } \partial\Omega \end{cases}$$

then

$$\liminf_{R\to+\infty} \frac{\sup_{\{x\in\Omega:|x|=R\}}|u-v|}{\log R} > 0. \tag{1}$$

In particular, if Ω is an exterior domain and the corresponding graph Σ_u is minimal, i.e. $\mathcal{H}_u \equiv 0$, with boundary located in a horizontal plane $\partial\Sigma_u \subset \mathbb{R}^2 \times \{c\}$, then the height function u satisfies

$$\liminf_{R\to+\infty} \frac{\max_{\{x\in\mathbb{R}^2:|x|=R\}}|u|}{\log R} > 0, \tag{2}$$

unless $\Sigma_u \subset \mathbb{R}^2 \times \{c\}$. Actually, it is well known that more is true: indeed, the exterior minimal graph has a limiting (say vertical) normal at infinity and one has the asymptotic behaviour $u(x) \sim a \log|x|$ where $a \in \mathbb{R}$ is called the logarithmic growth rate of u; [1, 13].

In recent years, Collin-Krust growth estimates (1) and (2) have been generalized in various ways and extended to several different settings; see e.g. [3, 6, 7, 8, 10]. In [10], the case of graphs over unbounded domains inside a generic complete, non-compact Riemannian manifold (M, g) is considered. Let $B_r(o)$ and $\partial B_r(o)$ denote, respectively, the intrinsic metric ball and sphere of (M, g) centered at $o \in M$ and of radius $r > 0$. Let also vol_{m-1} be the $(m-1)$-dimensional Hausdorff measure induced by the Riemannian volume vol. Under the *area growth* assumption

$$\int^{+\infty} \frac{dr}{\operatorname{vol}_{m-1}(\Omega \cap \partial B_r)} = +\infty \tag{3}$$

it is proved that if $u, v \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ are solutions of the problem

$$\begin{cases} \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) \leq \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) & \text{on } \Omega \\ v \geq u & \text{on } \partial\Omega \end{cases}$$

such that

$$\liminf_{R\to+\infty} \frac{\sup_{\partial B_R\cap\Omega}(u-v)}{\int_{R_0}^{R} \frac{dr}{\operatorname{vol}_{m-1}(\Omega\cap\partial B_r)}} = 0,$$

then

$$v \geq u \quad \text{on } \Omega.$$

As a consequence, if Σ_u is an exterior graph in $M \times \mathbb{R}_{\geq c}$ with non-positive mean curvature $\mathcal{H}_u \leq 0$ and with boundary $\partial\Sigma_u \subseteq M \times \{c\}$ we get the lower height estimate

$$\liminf_{R\to+\infty} \frac{\max_{\partial B_R} u}{\int_{R_0}^{R} \frac{dr}{\operatorname{vol}_{m-1}\partial B_r}} > 0, \tag{4}$$

unless $\Sigma_u \subset M \times \{c\}$. This last conclusion appears also in the influential paper [12] devoted to Liouville-type results for sub/supersolutions of nonlinear elliptic operators in divergence form. It is worth to remark that the area growth condition (3) implies that the underlying manifold with boundary Ω is recurrent with respect to the reflecting Brownian motion. If Ω is an exterior domain then the base manifold M is itself recurrent (equivalently, in potential theoretic terms, M is parabolic); see [5].

In the present paper, using a simple comparison argument, we shall provide an upper estimate for the growth rate of the height function u of an exterior graph $\Sigma_u \subset M \times \mathbb{R}$ of non-negative mean curvature $\mathcal{H}_u \geq 0$. We could speak of a "reverse Collin-Krust estimate" that complements (4) and, in some sense, completes the picture. Recall that o is a pole of (M, g) if the corresponding exponential map $\exp_o : T_oM \to M$ is a global diffeomorphism.

Theorem A. *Let (M, g) be a complete m-dimensional manifold with a pole $o \in M$ and radial sectional curvature*

$$\mathrm{Sect}^M_{\mathrm{rad}} \leq -\delta,$$

with $\delta \geq 0$. Consider the exterior upper half-catenoid

$$\Gamma_{a,\delta} = \{(x, \mathrm{C}_{a,\delta}(x)) : x \in M \setminus B_a(o)\} \hookrightarrow M \times \mathbb{R}$$

where

$$\mathrm{C}_{a,\delta}(x) = \int_a^{r(x)} \frac{g_\delta^{m-1}(a)}{\sqrt{g_\delta^{2(m-1)}(s) - g_\delta^{2(m-1)}(a)}} ds \tag{5}$$

and

$$g_\delta(r) = \begin{cases} r & \text{if } \delta = 0, \\ \frac{1}{\sqrt{\delta}} \sinh(\sqrt{\delta} r) & \text{if } \delta > 0. \end{cases}$$

Let

$$\Sigma_u = \{(x, u(x)) : x \in M \setminus B_a(o)\} \hookrightarrow M \times \mathbb{R}$$

be an exterior graphical hypersurface whose mean curvature satisfies

$$\mathcal{H}_u \geq 0.$$

If

$$\partial \Sigma_u \subset M \times \{0\}, \tag{6}$$

then the height function u satisfies

$$\limsup_{r \to +\infty} \frac{\min_{\partial B_r(o)} u}{\mathrm{C}_{a,\delta}(r)} \leq 1. \tag{7}$$

In particular, Σ_u cannot be contained in the region of $M \times \mathbb{R}$ above $(1+\epsilon)\Gamma_{a,\delta}$, for any $\epsilon > 0$. Moreover, if

$$\int^{+\infty} \frac{dr}{\mathrm{vol}_{m-1}(\partial B_r)} = +\infty,$$

the estimate (7) *holds without requiring the boundary condition* (6)*.*

Minimal catenoids in the spaceforms $\mathbb{R}^m$ and $\mathbb{H}^m$ show that the estimate is essentially sharp; see also Remark 3.1. Actually, we will be able to prove a version of Theorem A in the much more general setting of non-linear elliptic PDE's satisfying certain structural conditions and assuming that the solution is defined on an unbounded domain of M containing the interior ball; see Remark 4.3b at the end of the paper.

As it is already visible in the statement of Theorem A, unlike Collin-Krust type results, the upper growth rate is not directly related to the area growth of the domain but, instead, it is related to "catenoid-like" functions adapted to the nonlinear operator at hand. It is a pure chance that, for 2-dimensional Euclidean graphs, the growth rate of this functions compares with the area growth appearing in (2) and (4). In this setting, it could be interesting to point out the following simple consequence for 2-dimensional radial graphs.

Corollary B. *Let* $(M,g)=(\mathbb{R}^2,\lambda^2(x)\mathrm{can})$ *be a conformal Riemann surface whose Gaussian curvature* $K(x)$ *satisfies*

$$-k(r(x))\leq K(x)\leq 0,$$

where $r(x)=\mathrm{dist}(x,o)$ *and* $0\leq k(t)\in C^0(\mathbb{R})$ *is such that*

$$s\cdot k(s)\in L^1(+\infty).$$

Let $u\in C^2((a,+\infty))\cap C^0([a,+\infty))$ *define a radial graph*

$$\Sigma_u=\{(x,u(r(x))):x\in M\setminus B_a(o)\}$$

with mean curvature $\mathcal{H}_u\geq 0$*. Then*

$$\frac{u(r(x))}{\log(r(x))}\leq C,\ \ r(x)\gg 1, \tag{8}$$

for some constant $C>0$*.*

Remark 0.1. The conclusion of Corollary B compares with the classical logarithmic growth rate of exterior minimal graphs over $\mathbb{R}^2$. Actually, as it will be clear from the proof, when the exterior radial graph Σ_u is minimal and it is contained, say, in the half-space $\Sigma_u\subset M\times\mathbb{R}_{\geq 0}$ with boundary $\partial\Sigma_u\subset M\times\{0\}$, then either $\Sigma_u\subset M\times\{0\}$ or

$$u(r(x))\asymp\log(r(x)),\ \ r(x)\gg 1.$$

This follows by combining (8) with the Collin-Krust type estimate (4).

Proof. By the Cartan-Hadamard theorem each point $o\in M$ is a pole. On the other hand, by a result of R. Greene and H.H. Wu, [4, Theorem 4.C], (M,g) is bi-Lipschitz equivalent to $\mathbb{R}^2$, therefore we have

$$\mathrm{vol}B_r^M(o)\asymp\mathrm{vol}B_r^{\mathbb{R}^2}(0)\asymp r^2$$

which, in turn, implies

$$\int^{+\infty} \frac{dr}{\mathrm{vol}_{m-1}(\partial B_r^M(o))} = +\infty.$$

To conclude the validity of (8), recall that $m = 2$ and observe that in the definition (5) of $\mathrm{C}_{a,0}$ we have $g_0(r) = r$. Therefore, $\mathrm{C}_{a,0}(x) \sim a \log r(x)$, as $r(x) \to +\infty$.

□

The paper is organized as follows. First, we prove a general comparison principle on annuli that extends to supersolutions of non-linear elliptic PDEs a classical result by R. Osserman valid for minimal graphs in $\mathbb{R}^3$. An abstract version of this result is obtained in Proposition 2.2 and a concrete realization, involving "catenoid-like" functions on non-positively curved manifolds with a pole, is stated in Theorem 4.1. The comparison principle is then applied to get information on the growth of supersolutions over exterior domains: in Theorem 4.2 we consider the case of non-positive, interior boundary conditions and in the subsequent Corollary 4.4 we point out how the interior boundary condition can be avoided in parabolic situations.

1. The φ-Laplacian

Let (M, g) be a smooth, connected, non-compact, complete Riemannian manifold of dimension $m \geq 2$. According to the terminology introduced in [12], we put the following

Definition 1.1. *The φ-Laplacian of a function $u \in W^{1,p}_{\mathrm{loc}}$ is the nonlinear, divergence-form operator defined by*

$$\mathcal{L}_\varphi(u) = \mathrm{div}\left(|\nabla u|^{-1} \varphi\left(|\nabla u|\right) \nabla u\right),$$

where $\varphi \in C^0\left([0, +\infty)\right) \cap C^1\left((0, +\infty)\right)$ satisfies the following structural conditions:

$$\text{(SC)} \qquad (i)\ \varphi(0) = 0,\ (ii)\ \frac{d\varphi}{dt}(t) > 0\ \forall t > 0,\ (iii)\ \varphi(t) \leq A t^{p-1},$$

for some constants $A, p > 1$.

Remark 1.2. Usually, condition (ii) is replaced by the less stringent assumption that $\varphi(t) > 0$ for $t > 0$. However, in view of our purposes, we have preferred to incorporate in the definition the strict monotonicity of φ. This property is needed for a comparison theory to work.

Remark 1.3. The φ-Laplacian encloses very different type of non-linearities. We limit ourselves to point out that the choice

$$\varphi(t) = \frac{t}{\sqrt{1 + t^2}},$$

gives rise to the mean curvature operator on the m-dimensional manifold (M, g).

Remark 1.4. Note that, even if the function u is assumed to be C^2, it may happen that the vector field $Z = |\nabla u|^{-1}\varphi(|\nabla u|)\nabla u$ is not C^1 at points where $|\nabla u| = 0$. Of course, when the aforementioned vector field is not C^1, then the divergence must be understood in the sense of distributions. More generally, given any two functions $u, v \in W^{1,p}_{\text{loc}}(\Omega)$ on a domain $\Omega \subset M$, the inequality

$$\mathcal{L}_\varphi(u) \geq \mathcal{L}_\varphi(v)$$

means that

$$-\int_\Omega \langle |\nabla u|^{-1}\varphi(|\nabla u|)\nabla u - |\nabla v|^{-1}\varphi(|\nabla v|)\nabla v, \nabla\psi\rangle \geq 0,$$

for all $0 \leq \psi \in C^\infty_c(\Omega)$.

The φ-Laplacian enjoys the usual comparison principle for continuous distributional solutions. In fact, we have the following result. See [11, 12].

Lemma 1.5. *Let (M, g) be an m-dimensional Riemannian manifold and let $\Omega \subset M$ be a bounded domain. Let $u, v \in W^{1,p}_{\text{loc}}(\Omega) \cap C^0(\bar{\Omega})$ be solutions of*

$$\begin{cases} \mathcal{L}_\varphi(v) \geq \mathcal{L}_\varphi(u) & \text{in } \Omega \\ v \leq u & \text{on } \partial\Omega, \end{cases}$$

Then $v \leq u$ in $\bar{\Omega}$.

Remark 1.6. The continuity of u and v up to the boundary can be omitted, and the conclusion will hold a.e. in Ω, as soon as one gives an appropriate interpretation to the boundary condition. For instance, according to [11, Theorem 3.4.1], the comparison principle holds for $u, v \in W^{1,p}_{\text{loc}}(\Omega)$ provided $v \leq u$ on $\partial\Omega$ means that, for every $\delta > 0$, there exists a neighborhood of $\partial\Omega$ where $v \leq u + \delta$ a.e. This allows one to include e.g. the case of infinite boundary values.

For later purposes we also recall the following terminology from [12]

Definition 1.7. *The Riemannian manifold (M, g) is φ-parabolic if the only solutions $u \in W^{1,p}_{\text{loc}}(M) \cap C^0(M)$ of $\mathcal{L}_\varphi(u) \geq 0$ satisfying $\sup_M u < +\infty$ are the constant functions.*

It turns out that, for a geodesically complete manifold (M, g), φ-parabolicity is related to volume growth properties of the space. Namely, it is proved in [12] that a complete Riemannian manifold is φ-parabolic provided

$$\int^{+\infty} \frac{dr}{\text{vol}_{m-1}(\partial B_r)^{\frac{1}{p-1}}} = +\infty. \tag{9}$$

If $\varphi(t) = t$ then $\mathcal{L}_\varphi$ reduces to the Laplace-Beltrami operator of (M, g) and φ-parabolicity is nothing but the usual notion of parabolicity from the linear potential theory. In this case, since $p = 2$, condition (9) reduces the classical area-growth condition first discovered by L.V. Ahlfors.

Remark 1.8. It is interesting to observe that, even for the mean curvature operator we have $p = 2$ and, therefore, the volume growth related to the linear Laplace-Beltrami operator covers also the non-linear potential theory of the mean curvature operator.

2. An abstract comparison principle on annuli

Inspired by a comparison principle due to R. Osserman for exterior minimal graphs in the Euclidean space, [9, Lemma 10.2], and making an essential use of Lemma 1.5, we prove the following abstract result.

Notation 2.1. From now on, given real numbers $b > a > 0$ and a point $o \in M$ we denote by $\mathcal{A}_{a,b}(o)$ the open annulus $B_b(o) \setminus \bar{B}_a(o)$.

Proposition 2.2. *Let (M, g) be a complete Riemannian manifold and let o be a reference point. Let $0 < a < b$ be real numbers such that $a < r_{\mathrm{inj}}(o)$. Consider the one-parameter family of functions*

$$\{v_\lambda\}_{\lambda \in [1,b/a)} \subset C^1\big(\mathcal{A}_{\lambda a,b}(o)\big) \cap C^0\big(\bar{\mathcal{A}}_{\lambda a,b}(o)\big)$$

satisfying the following conditions:

$$\begin{aligned} &a)\ \mathcal{L}_\varphi(v_\lambda) \geq 0 \quad on\ \mathcal{A}_{\lambda a,b}(o), \\ &b)\ \partial_r v_\lambda(x) \to +\infty\ \ as\ x \to \partial B_{\lambda a}(o); \\ &c)\ v_\lambda(x) \to v_1(x)\ \ as\ \lambda \to 1^+, \end{aligned} \tag{10}$$

where ∂_r denotes the radial derivative with respect to o, and the convergence in c) is pointwise on $\mathcal{A}_{a,b}(o)$ and uniformly on $\partial B_b(o)$. If

$$u \in C^1\big(\mathcal{A}_{a,b}(o)\big) \cap C^0\big(\bar{\mathcal{A}}_{a,b}(o)\big)$$

is a solution of the problem

$$\begin{cases} \mathcal{L}_\varphi(u) \leq 0 \leq \mathcal{L}_\varphi(v_1) & \text{in } \mathcal{A}_{a,b}(o) \\ u \geq v_1 & \text{on } \partial B_b(o), \end{cases} \tag{11}$$

then

$$u \geq v_1 \quad \text{on } \bar{\mathcal{A}}_{a,b}(o).$$

Remark 2.3. As it will be clear from the proof, the C^1 regularity of u and v is required only in a neighborhood of the interior sphere $\partial B_a(o)$.

Proof. Let $\lambda \in (1, b/a)$ and set

$$\varepsilon_\lambda := \max_{\partial B_b(o)} |v_\lambda(x) - v_1(x)|.$$

Thus, for all $x \in \partial B_b(o)$,

$$v_\lambda(x) - u(x) = (v_\lambda(x) - v_1(x)) - (u(x) - v_1(x)) \leq \varepsilon_\lambda.$$

We claim that

$$v_\lambda - u \leq \varepsilon_\lambda \quad \text{on } \partial B_{\lambda a}(o). \tag{12}$$

Since, in this case,

$$\begin{cases} \mathcal{L}_\varphi(u+\varepsilon_\lambda) \le \mathcal{L}_\varphi(v_\lambda) & \text{in } \mathcal{A}_{\lambda a,b}(o) \\ u+\varepsilon_\lambda \ge v_\lambda & \text{on } \partial\mathcal{A}_{\lambda a,b}(o), \end{cases}$$

the required conclusion follows immediately from Lemma 1.5 and by letting $\lambda \to 1^+$.

In order to prove (12) we reason by contradiction and assume that

$$M_\lambda := \max_{\partial B_{\lambda a}(o)} (v_\lambda - u) > \varepsilon_\lambda.$$

Then,

$$(v_\lambda - u) \le M_\lambda \text{ on } \partial\mathcal{A}_{\lambda a,b}(o),$$

and using Lemma 1.5 once more we get

$$(v_\lambda - u) \le M_\lambda \quad \text{on } \bar{\mathcal{A}}_{\lambda a,b}(o). \tag{13}$$

Take $x_\lambda \in \partial B_{\lambda a}(o)$ such that

$$M_\lambda = (v_\lambda - u)(x_\lambda),$$

and consider a unit-speed, minimizing geodesic γ issuing from o and passing at time $t = \lambda a$ through x_λ. Since $0 < a < r_{\text{inj}}(o)$, we can assume that λ is so close to 1 that $\lambda a < r_{\text{inj}}(o)$. In particular, the distance function $r(x) = d(x,o)$ is smooth along γ on $[0, \lambda a + \epsilon]$, $0 < \epsilon \ll 1$, and on the same interval, $\nabla r(\gamma(t)) = \dot\gamma(t)$. Consider the function

$$f(t) = v_\lambda \circ \gamma(t) - u \circ \gamma(t)$$

on $[0, \lambda a + \epsilon]$. Since u has finite gradient along $\partial B_{\lambda a}(o)$ we have

$$\frac{df}{dt}(t) = (\partial_r v_\lambda)(\gamma(t)) - (\partial_r u)(\gamma(t)) \to +\infty \text{ as } t \to (\lambda a)^+,$$

and this implies that $f(t)$ is strictly increasing on $(\lambda a, \lambda a + \epsilon]$ if ϵ is small enough. Since $f(\lambda a) = M_\lambda$ we deduce that $f(\lambda a + \epsilon) > M_\lambda$, proving that at the point $x_\epsilon = \gamma(\lambda a + \epsilon) \in \partial B_{\lambda a+\epsilon}(o)$ it holds

$$(v_\lambda - u)(x_\epsilon) > M_\lambda.$$

This contradicts (13) and completes the proof of the Proposition. □

3. Catenoid-like functions

Clearly, the main point of the previous Lemma is the existence of the one-parameter family of functions v_λ. We are going to show that this family can be explicitly constructed on any complete, m-dimensional Riemannian manifold (M, g) with a pole $o \in M$ and satisfying

$$\text{Sect}^M_{\text{rad}} \le -\delta, \tag{14}$$

for some constant $\delta \ge 0$, provided the φ-Laplacian enjoys the further structural condition:

$$iv) \lim_{t\to+\infty} \varphi(t) = L < +\infty. \tag{SC}$$

Indeed, suppose we are in this situation. Clearly we can always assume $L = 1$. Let $g_\delta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be the smooth solution of the Cauchy problem

$$\begin{cases} g_\delta'' - \delta g_\delta = 0 \\ g_\delta(0) = 0, \; g_\delta'(0) = 1, \end{cases}$$

that is

$$g_\delta(r) = \begin{cases} r & \text{if } \delta = 0, \\ \frac{1}{\sqrt{\delta}} \sinh(\sqrt{\delta} r) & \text{if } \delta > 0. \end{cases}$$

Next, fix $a > 0$ and define the smooth radial function

$$\mathrm{C}_{\varphi,a,\delta} : M \setminus B_a(o) \to \mathbb{R}_{>0}$$

as

$$\mathrm{C}_{\varphi,a,\delta}(r(x)) := \int_a^{r(x)} \varphi^{-1} \left(\frac{g_\delta^{m-1}(a)}{g_\delta^{m-1}(s)} \right) ds. \tag{15}$$

Remark 3.1. We stress that the structural condition (SC) iv) is satisfied when $\mathcal{L}_\varphi$ is the mean curvature operator. In this case, if we specify M to be one of the space-forms $\mathbb{R}^m$ or $\mathbb{H}^m$, the exterior graphs

$$\Gamma_{a,\delta} = \{(x, \mathrm{C}_{\varphi,a,\delta}(x)) : x \in M \setminus B_a(o)\}$$

are exactly the standard *upper catenoids* of mean curvature $\mathcal{H} = 0$.

Now, we consider the family of rescaled radial functions

$$v_\lambda(x) = \lambda \mathrm{C}_{\varphi,a,\delta}(\lambda^{-1} r(x)),$$

with $\lambda \geq 1$. In view of (14), we can apply the Laplacian Comparison Theorem and compute

$$\begin{aligned} \mathcal{L}_\varphi(v_\lambda) &= \mathrm{div} \left(\varphi \left(\frac{d\mathrm{C}_{\varphi,a,\delta}}{dr}(\lambda^{-1} r) \right) \nabla r \right) \\ &\geq g_\delta^{m-1}(a) \cdot \left\{ \frac{(m-1)}{g_\delta^{m-1}(\lambda^{-1} r)} \left(\frac{g_\delta'(r)}{g_\delta(r)} - \frac{1}{\lambda} \frac{g_\delta'(\lambda^{-1} r)}{g_\delta(\lambda^{-1} r)} \right) \right\}. \end{aligned}$$

From the expression

$$\frac{g_\delta'(r)}{g_\delta(r)} = \begin{cases} \frac{1}{r} & \text{if } \delta = 0, \\ \sqrt{\delta} \operatorname{cotanh}(\sqrt{\delta} r) & \text{if } \delta > 0 \end{cases}$$

it is easy to verify that

$$\frac{g_\delta'(r)}{g_\delta(r)} - \frac{1}{\lambda} \frac{g_\delta'(\lambda^{-1} r)}{g_\delta(\lambda^{-1} r)} \geq 0$$

and, therefore,

$$\mathcal{L}_\varphi(v_\lambda) \geq 0.$$

Thus, (10) a) is satisfied. Next, we note that, according to (SC) iv),

$$\begin{aligned}\partial_r v_\lambda(x) &= \frac{d\mathrm{C}_{\varphi,a,\delta}}{dr}(\lambda^{-1}r(x)) \\ &= \varphi^{-1}\left(\frac{g_\delta^{m-1}(a)}{g_\delta^{m-1}(\lambda^{-1}r(x))}\right) \\ &\to +\infty, \text{ as } r(x) \to \lambda a,\end{aligned}$$

and this proves the validity of (10) b). Finally, condition (10) c) is satisfied because, for every fixed $a < r \leq b$,

$$\max_{\partial B_r(o)} |v_\lambda - v_1| = |\lambda \mathrm{C}_{\varphi,a,\delta}(\lambda^{-1}r) - \mathrm{C}_{\varphi,a,\delta}(r)| \to 0, \text{ as } \lambda \to 1.$$

Remark 3.2. Actually, the construction of the one-parameter family of functions v_λ can be carried out in a more general setting. Namely, suppose we are given a smooth even function $G : \mathbb{R} \to \mathbb{R}_{\geq 0}$ and let (M, g) be an m-dimensional Riemannian manifold with a pole o whose radial sectional curvature satisfies

$$\mathrm{Sect}^M_{\mathrm{rad}} \leq -G(r). \tag{16}$$

Let $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be the unique solution of

$$\begin{cases} g'' - Gg = 0 \\ g(0) = 0, \ g'(0) = 1. \end{cases} \tag{17}$$

and assume that

$$r \mapsto \log g(r) \text{ is a concave function.} \tag{18}$$

Then, up to defining

$$\mathrm{C}_{\varphi,a,g}(x) := \int_a^{r(x)} \varphi^{-1}\left(\frac{g^{m-1}(a)}{g^{m-1}(s)}\right) ds, \tag{19}$$

the corresponding family of dilated functions v_λ satisfies all the conditions stated in (10). The verifications are completely similar to the above and left to the reader.

4. On the growth of supersolutions on manifolds with a pole

Using in Proposition 2.2 the "catenoid-like" functions constructed in the previous section, we deduce the validity of the next concrete comparison result.

Theorem 4.1. *Let (M, g) be an m-dimensional complete Riemannian manifold with a pole o and with radial sectional curvature satisfying* (16) – (18). *Let $0 < a < R$ be real numbers and assume that the φ-Laplace operator*

$\mathcal{L}_\varphi$ satisfies the structural conditions (SC), *i)–iv). If $u \in C^1(\mathcal{A}_{a,R}(o)) \cap C^0(\bar{A}_{a,R}(o))$ solves*

$$\begin{cases} \mathcal{L}_\varphi(u) \le 0 & \text{in } \mathcal{A}_{a,R}(o) \\ u \ge \mathrm{C}_{\varphi,a,g} & \text{on } \partial B_R(o) \end{cases} \tag{20}$$

then

$$u \ge \mathrm{C}_{\varphi,a,g} \quad \text{on } \bar{\mathcal{A}}_{a,R}(o),$$

where $\mathrm{C}_{\varphi,a,g}$ *is the catenoid-like function defined in* (19)*.*

We are now able to prove the following result concerning the growth of solutions of general nonlinear PDEs over exterior domains.

Theorem 4.2. *Let (M, g) be an m-dimensional manifold with a pole $o \in M$ whose radial sectional curvature satisfies conditions* (16)–(18)*. Let $u \in C^1\left(M \setminus \bar{B}_a(o)\right) \cap C^0\left(M \setminus B_a(o)\right)$, $a > 0$, be a solution of the problem*

$$\begin{cases} \mathcal{L}_\varphi(u) \le 0 & \text{on } M \setminus \bar{B}_a(o) \\ u \le 0 & \text{on } \partial B_a(o) \end{cases}$$

where the φ-Laplace operator $\mathcal{L}_\varphi$ satisfies the structural conditions (SC)*, i)–iv). Then*

$$\limsup_{r \to +\infty} \frac{\min_{\partial B_r(o)} u}{\mathrm{C}_{\varphi,a,g}(r)} \le 1, \tag{21}$$

where $\mathrm{C}_{\varphi,a,g}$ *is the catenoid-like function defined in* (19)*.*

Proof. By contradiction, suppose that (7) is not satisfied. Then, along a sequence $r_n \nearrow +\infty$ it holds

$$\min_{\partial B_{r_n}(o)} u \ge (1+\eta)\mathrm{C}_{\varphi,a,g}(r_n),$$

for some $\eta > 0$. Let

$$C^* := \lim_{n \to +\infty} \mathrm{C}_{\varphi,a,g}(r_n) \le +\infty,$$

and define

$$c = \min\left\{1, \frac{\eta C^*}{4}\right\}.$$

Then, for $n \gg 1$,

$$\min_{\partial B_{r_n}(o)} (u - c) \ge \left(1 + \frac{\eta}{2}\right) \mathrm{C}_{\varphi,a,g}(r_n).$$

It follows that u satisfies

$$\begin{cases} \mathcal{L}_\varphi(u-c) \le 0 \le \mathcal{L}_\varphi(\mathrm{C}_{\varphi,a,g}) & \text{in } \mathcal{A}_{a,r_n}(o) \\ u - c > \mathrm{C}_{\varphi,a,g} & \text{on } \partial B_{r_n}(o). \end{cases}$$

An application of Theorem 4.1 yields

$$u - c \ge \mathrm{C}_{\varphi,a,g} \quad \text{on } \partial B_a(o),$$

and this contradicts the fact that, regardless of the value of $c>0$,

$$u-c<\mathrm{C}_{\varphi,a,g}\quad \text{on } \partial B_a(o).$$

The proof of the theorem is completed.

□

Remarks 4.3. Inspection of the above proof shows that the result can be generalized and extended in the following directions.

4.3a. Exterior domains in φ-parabolic manifolds. In case $C^*=+\infty$, the boundary assumption $\max_{\partial B_a(o)} u \leq 0$ is not required. In this respect, note that the radial function $\mathrm{C}_{\varphi,a,\delta}$ is unbounded precisely when

$$\varphi^{-1}\left(\frac{g^{m-1}(a)}{g^{m-1}(r)}\right)\notin L^1(+\infty), \tag{22}$$

and this is related to the φ-parabolicity of the underlying Riemannian manifold. Indeed, if the radial curvature of the complete m-dimensional manifold (M,g) with a pole $o\in M$ satisfies

$$\mathrm{Sect}^M_{\mathrm{rad}}\leq -G(r)\leq 0,$$

then, as a consequence of the Bishop-Gromov comparison theorem, one has

$$\mathrm{vol}_{m-1}(\partial B_r(o))\geq c_m g^{m-1}(r),$$

where c_m is the volume of the unit sphere in $\mathbb{R}^m$ and g is the solution of the Cauchy problem (17). In particular:

$$\varphi^{-1}\left(\frac{g^{m-1}(a)}{g^{m-1}(r)}\right)\gtrsim \frac{1}{g^{\frac{m-1}{p-1}}(r)}\gtrsim \frac{1}{\mathrm{vol}_{m-1}(\partial B_r(o))^{\frac{1}{p-1}}}$$

where the first inequality is a consequence of the structural property (SC) iii). If we assume that

$$\int^{+\infty}\frac{dr}{\mathrm{vol}_{m-1}(\partial B_r(o))^{\frac{1}{p-1}}}=+\infty, \tag{23}$$

and hence (M,g) is φ-parabolic, then (22) is satisfied. We thus deduce the validity of the next result.

Corollary 4.4. *Let (M,g) be an m-dimensional manifold with a pole $o\in M$ and satisfying the volume growth property* (23)*. Assume also that the radial sectional curvature of M satisfies conditions* (16) – (18)*. If*

$$u\in C^1\left(M\setminus \bar{B}_a(o)\right)\cap C^0\left(M\setminus B_a(o)\right),$$

is a solution of $\mathcal{L}_\varphi(u)\leq 0$ on $M\setminus \bar{B}_a(o)$ then

$$\limsup_{r\to+\infty}\frac{\min_{\partial B_r(o)} u}{\mathrm{C}_{\varphi,a,g}(r)}\leq 1,$$

where $\mathrm{C}_{\varphi,a,\delta}$ is the catenoid-like function defined in (19)*.*

4.3b. Supersolutions over more general domains. The use of the comparison principle discussed in Remark 1.6 allows us to obtain a growth estimate for supersolutions with infinite boundary values over domains that look like exterior domains in a manifold with boundary. More precisely:
Keeping the notation and the assumptions on the pointed Riemannian manifold (M, g, o), assume that $\Omega \subseteq M$ is an unbounded domain such that $B_a(o) \Subset \Omega$. If $u \in C^1\left(\Omega \setminus \bar{B}_a(o)\right)$ is a solution of the problem

$$\begin{cases} \mathcal{L}_\varphi(u) \leq 0 & \text{on } \Omega \setminus \bar{B}_a(o) \\ u \leq 0 & \text{on } \partial B_a(o) \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

then

$$\limsup_{r \to +\infty} \frac{\min_{\partial B_r(o) \cap \Omega} u}{\mathrm{C}_{\varphi,a,g}(r)} \leq 1.$$

In case $\Omega = M$, hence $\partial\Omega = \emptyset$, the further boundary condition is assumed as to be trivially satisfied and we recover the original version of the result.

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