Polynomial Sampling and Fractal Measures:
I.F.S.–Gaussian Integration *

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Abstract. Measures generated by Iterated Function Systems can be used in place of atomic measures in Gaussian integration. A stable algorithm for the numerical solution of the related approximation problem—an inverse problem in fractal construction—is proposed.

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Dedicated to Walther Gautschi

1. Introduction

Constructing the Jacobi matrix of a system of orthogonal polynomials leads naturally to an approximation of the orthogonality measure by a sum of atomic measures. Despite the fact that the numerical computation of the Jacobi matrix may require considerable theoretical and numerical effort (Gau, 2004) and despite the marvelous convergence properties of Gaussian integration, this approximation is rather crude, especially when the sampled measure is singular continuous (Man, 1996). This limitation can be overcome by finding systems of iterated functions with probabilities (I.F.S.) (Hut, 1981; B-D, 1986) and the associated invariant measures so to integrate exactly polynomials of given order: this defines a generalized I.F.S.-Gaussian integration that will be described in this paper.

I adopt a variant of this approximation problem—working with I.F.S. made of affine maps with equal contraction ratios (E-Y, 1989; H-M, 1990; B-D, 1991) for which I present here, for the first time to the best of my knowledge, a stable solution algorithm. The theory is accompanied with illustrative numerical examples.

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2. Measures, Iterated Function Systems and Orthogonal Polynomials

Let $\mu$ be a normalized positive Borel measure on the unit interval: $\mu \in M_1([0,1])$. The set of orthogonal polynomials of $\mu$, $\{p_n(\mu,s)\}_{n\in\mathbb{N}}$, is defined by:

$$\int d\mu(s) \; p_n(\mu,s) \; p_m(\mu,s) = \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker’s delta. They fulfill a three-terms relation:

$$s \; p_n(\mu,s) = a_n \; p_n(\mu,s) + b_{n+1} \; p_{n+1}(\mu,s) + b_n \; p_{n-1}(\mu,s).$$  \hspace{1cm} (1)

The two infinite sequences of coefficients $a_n$ and $b_n \geq 0$ uniquely encode the measure $\mu$, in force of Favard’s theorem. The beginning of the sequence of $b_n$’s is composed of strictly positive entries: if a particular $n$ exists such that $b_n = 0$, the sequence terminates at that value and the corresponding measure is discrete, composed of a finite number of atoms. This case is of particular relevance for Gaussian integration. In fact, let the $a_n$’s be the diagonal and $b_n$’s the leading sub-diagonal of the symmetric Jacobi matrix $J_\mu$,

$$J_\mu := \begin{pmatrix} a_0 & b_1 & b_2 & \ldots \\ b_1 & a_1 & b_2 & \ldots \\ b_2 & a_2 & b_3 & \ldots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}. \hspace{1cm} (2)$$

Let now $J_\mu^n$ be the principal minor of rank $n$ of $J_\mu$, that can also be obtained setting $b_n = 0$. It is well known (G-W, 1989) that Gaussian points are the eigenvalues $\{x_k\}_{k=1}^n$ of $J_\mu^n$. $J_\mu^n x_k = x_k y_k$ and Gaussian weights, $\{w_k\}_{k=1}^n$, are the squares of the first components of the normalized eigenvectors $\{y_k\}_{k=1}^n$. Gaussian integration can be seen as an approximation of the measure $\mu$ by the discrete, $n$-point measures $\mu^n$:

$$\mu^n := \sum_{k=1}^n w_k \delta_{x_k}, \hspace{1cm} (3)$$

where $\delta_x$ is the atomic measure at $x$. Despite the remarkable convergence properties of Gaussian integration, the approximation obtained from (3) is rather poor, when a characterization of the multifractal properties (Pes, 1996) of the sampled measure $\mu$ is sought. We then select a different family of measures as approximation tool: the balanced measures of Iterated Function Systems (Hut, 1981; B-D, 1986). In its simplest form, an I.F.S. is a finite collection of affine, contractive maps of the unit interval into itself:

$$\phi_j(s) = \delta_j(s - \beta_j) + \beta_j, \; j = 1, \ldots, M. \hspace{1cm} (4)$$

The parameters $\{\delta_j\}_{j=1}^M$, $0 \leq \delta_j < 1$ can be described as contraction ratios, since the unit interval is mapped by $\phi_j$ into a subinterval of length $\delta_j$. The
fixed points of the maps, \( \{ \beta_j \}_{j=1}^M \), are the second set of parameters specifying an I.F.S.. An iterative construction of a fractal set is obtained by mapping \([0, 1]\) by the collection of I.F.S. maps into \( \{ \phi_j([0, 1]) \}_{j=1}^M \) and then into

\[
\{ \phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_k} \circ \phi_{\sigma_1}([0, 1]) \}
\]

with \( \sigma_n \in \{ 1, \ldots, M \} \) for \( n = 1, \ldots, k \). Take the union of these intervals and let \( k \) tend to infinity. This construction tends (in the Hausdorff metric) to the attractor of the I.F.S., that is, the set \( A \) that satisfies the equation

\[
A = \bigcup_{j=1}^M \phi_j(A).
\]

A family of measures on \( A \) can be constructed in terms of a third set of parameters, the probabilities \( \{ \pi_j \}_{j=1}^M \), \( \pi_j > 0 \), \( \sum_j \pi_j = 1 \). Let the uniform Lebesgue measure be defined on \([0, 1]\), and let us add to the iterative construction of the I.F.S. attractor just described the further specification that the Lebesgue measure on \( \phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_k} \circ \phi_{\sigma_1}([0, 1]) \) is multiplied by \( \pi_{\sigma_1} \cdots \pi_{\sigma_k} \pi_{\sigma_1} \). This construction defines a limit measure on \([0, 1]\) that satisfies, and can be equally defined via, the equation

\[
\int f(s) d\mu(s) = \sum_{j=1}^M \pi_j \int (f \circ \phi_j)(s) d\mu(s),
\]

required to hold for any continuous function \( f \). Since I.F.S. measures are supported on a compact set, the associated moment problem admits a unique solution and the orthogonal polynomials \( p_n(\mu; s) \) are defined for all orders.

Among all affine I.F.S. we shall consider in this paper a sub-class that is particularly convenient for developing an approximation theory: we suppose that all contraction ratios are equal: \( \delta_j = \delta, \ j = 1, \ldots, M \). We shall call this a \( \delta \) – homogeneous affine I.F.S. (\( \delta \)-H.I.F.S.). Notice that discrete measures with finitely many atoms are a particular case of this family, that of maps with null contraction ratios: the measure in eq. (3) can be obtained by \( n \) maps with \( \delta_j = 0, \beta_j = x_j \) and \( \pi_j = w_j \), for \( j = 1 \) to \( n \).

### 3. An I.F.S. Approximation Problem

We can now introduce our basic approximation problem: *given an arbitrary measure \( \mu \in \mathcal{M}_1([0, 1]) \), find a \( \delta \)-H.I.F.S. and its balanced measure \( \mu' \), so that the Jacobi matrices of \( \mu \) and \( \mu' \) coincide up to size \( N \):

\[
J^N_\mu = J^N_{\mu'}.
\]

It is clear that (6) is equivalent to say that the first \( 2N \) moments of \( \mu \) and \( \mu' \) are equal. Indeed, moments were employed in the original formulation of this problem. Since the measure \( \mu' \) integrates exactly all polynomials up to degree \( 2N - 1 \) it can be properly termed an I.F.S.–Gaussian Approximation of \( \mu \).
It is not surprising that this problem may have infinite solutions, or no solution at all. Two main theoretical results describe this situation. The first is Elton and Yan’s theorem (E-Y, 1989):

**THEOREM 1.** Let $\delta > 0$. A solution of the problem 6 exists for all $N$ if and only if the function $\hat{\mu}(y)/\hat{\mu}(\delta y)$ is positive definite, where $\hat{\mu}(\cdot)$ is the Fourier transform of $\mu$.

A more flexible result is due to Handy and Mantica (H-M, 1990):

**THEOREM 2.** Let $\mu$ be a measure with an infinite number of points of increase. Then, for all $N > 0$ there exists $\delta_N > 0$ such that for all $\delta \in [0, \delta_N)$ the problem 6 has a unique solution with $M = N$.

**Sketch of the proofs of Thms. 1 and 2.** Fix the value of $\delta$, so that the remaining free parameters are $\beta_j$ and $\pi_j$, for $j = 1$ to $M$, where of course we are also free to choose the number of maps, $M$. Let us now encode these parameters in the discrete measure $\sigma$:

$$\sigma = \sum_{j=1}^{M} \pi_j \delta \beta_j.$$  \hfill (7)

First of all, choosing $f(s) = e^{-iys}$ in eq. (5) it is easy to see that the Fourier transform of $\sigma$ and $\mu$ are related by $\hat{\mu}(y) = \hat{\sigma}(y)/\hat{\mu}(\delta y)$, whence Thm. 1 follows.

Successively, choose $f(s) = s^k$ in eq. (5), to link the moments of $\mu$ and $\sigma$:

$$\mu_k = \sum_{j} \pi_j \int d\mu(s)(\phi_j(s))^k = \sum_{j} \pi_j (\delta s + \bar{\delta} \beta_j)^k = \sum_{l=0}^{k} B(k, l) \delta^l \mu_l \delta^{k-l} \sigma_{k-l},$$  \hfill (8)

where we have put $\bar{\delta} = 1 - \delta$ and where $B(k, l)$ is a binomial coefficient. Equations (8) form a triangular set of relations that permit to derive $\{\sigma_l\}_{l=0}^k$ from $\{\mu_l\}_{l=0}^k$ and vice versa. Therefore, if the I.F.S. parameters are known, one can easily compute the moments of $\sigma$ and en suite those of $\mu$.

Notice that in the inverse problem, when all moments $\{\mu_l\}_{l=0}^\infty$ are supposed to be known, one can in principle compute uniquely the full sequence $\{\sigma_l\}_{l=0}^\infty$, for any value of $\delta$. Clearly, given an arbitrary measure $\mu$, this sequence of values is not assured to be the sequence of moments of a positive measure $\sigma$, for otherwise one would violate Elton’s theorem. Hence, Handy and Mantica considered the truncated, finite sequences $\{\sigma_j\}_{j=0}^k$ and imposed Hankel-Hadamard inequalities, that are necessary and sufficient to ensure the existence of a such a measure. These determinantal inequalities are easily seen to be continuous functions of the contraction rate $\delta$. In addition, for $\delta = 0$ all inequalities are satisfied, since in this case $\mu$ and $\sigma$ coincide: this ensures the existence of the interval $[0, \delta_N)$ in Thm. 2.
The decoding problem of obtaining the map parameters as weights and locations of the atomic measures in eq. (7) from the moments \( \{\sigma_i\}_{i=0}^{2M} \) is known to be ill conditioned and the efforts of (E-Y, 1989; H-M, 1990) were bound to a very limited number of maps, as well as those of (B-D, 1991), who employed generalized moments, obtaining only a reduction of the coefficient in the exponential growth of the error. The problem was then basically abandoned at that point.

### 4. A Numerically Stable Stieltjes Solution

The ill-conditioning of the moment solution of the inverse fractal problem described in the previous section affected another problem: the construction of the Jacobi matrix of an I.F.S. measure. Standard techniques are in fact based on classical polynomial sampling: adapting the analysis of Gautschi (Gau, 1990) it appeared clearly (Man, 1996) that they cannot reliably sample a measure supported on a fractal set.

A different technique was needed and this was found (Man, 1996; Fis, 1997) see also (L-dV, 2006) avoiding the perilous passage through the moments’ mountain range. To further exploit this analogy with the magnificent ambiance of this meeting, one could say that the circumventing path was traced in a succession of demanding—but well conditioned mountain passes, like those encircling the Sella Group in the Dolomites. One of the key points of the technique in (Man, 1996) is a simple observation: adopt the compact notation \( \phi_\beta(s) \) for any of the I.F.S. maps,

\[
\phi_\beta(s) := \delta s + \tilde{\delta} \beta,
\]

where, as before, \( \tilde{\delta} := 1 - \delta \). Then, \( p_n(\mu; \phi_\beta(s)) \) is a polynomial of degree \( n \) in the variable \( s \), that can be expanded on the first \( n \) orthogonal polynomials. In this way, a direct technique of computing the Jacobi matrix associated with an I.F.S. measure was found.

In this section we describe a similar algorithm for the basic approximation problem introduced in Sect. 3. The algorithm takes as input the truncated Jacobi matrix \( J^N_\mu \) of an arbitrary measure \( \mu \in \mathcal{M}_1([0,1]) \), and a value of the real variable \( \delta \), and produces as output—when it exists—the truncated Jacobi matrix of the measure \( \sigma \). In the next section, the numerical stability of the algorithm is described: it is plausible, although not within the aims of the present paper, that a proof of this finding could be obtained following the lines of (Man, 1996).

Notice that the symmetrical rôle of \( \beta \) and \( s \) in the linear maps \( \phi \) implies that \( p_n(\mu; \phi_\beta(s)) \) is a homogeneous polynomial of degree \( n \) in the variables \( s \)
and $\beta$. This yields the double expansion
\[ p_n(\mu; \phi_{\beta}(s)) = \sum_{0 \leq k + r \leq n} \Gamma_{k,r}^n p_k(\mu; s) p_r(\sigma; \beta) \] (10)

in terms of the orthogonal polynomials of $\mu$ and of those of $\sigma$. The algorithm that we describe now is a recursive evaluation of the triangular matrices $\Gamma^n$ and of the Jacobi matrix $J_\sigma$. For simplicity of notation, in the following we shall let the indices $k$ and $r$ run freely, while assuming that $\Gamma_{k,r}^n = 0$ unless $0 \leq k + r \leq n$. Of fundamental importance are the following formulae:
\[ \phi_{\beta}(s) p_n(\mu; \phi_{\beta}(s)) = \sum_{k,r} \Gamma_{k,r}^n \left[ \delta B_k(\mu; s) p_r(\sigma; \beta) + \tilde{\delta} B_r(\sigma; \beta) p_k(\mu; s) \right], \] (11)

where we have put
\[ B_j(\cdot; s) := b_{j+1}(\cdot) p_{j+1}(\cdot; s) + a_j(\cdot) p_j(\cdot; s) + b_j(\cdot) p_{j-1}(\cdot; s) \] (12)

and $\cdot$ can be either $\mu$ or $\sigma$. To prove eq. (11) observe that
\[ \phi_{\beta}(s) p_n(\mu; \phi_{\beta}(s)) = (\delta s + \tilde{\delta} \beta) \sum_{k,r} \Gamma_{k,r}^n p_k(\mu; s) p_r(\sigma; \beta) \]

and the products $s p_k(\mu; s)$ and $\beta p_r(\sigma; \beta)$ can be dealt with using eq. (1).

We are now ready to introduce the algorithm, that runs in a cyclic sequence of three steps, described by the Lemmas below. Initialization: $n = 0$. It requires $\Gamma_{0,0}^0 = 1$, since $p_0(\mu; s) = 1 = p_0(\sigma; \beta)$ and $b_0(\mu) = b_0(\sigma) = 0$.

**Lemma 1.** The matrix element $a_n(\sigma)$ can be computed in terms of $\Gamma^n$ and linearly in terms of the Jacobi matrix entries $\{a_0(\mu), \ldots, a_n(\mu)\}$, $\{b_1(\mu), \ldots, b_n(\mu)\}$, $\{a_0(\sigma), \ldots, a_{n-1}(\sigma)\}$ and $\{b_1(\sigma), \ldots, b_n(\sigma)\}$.

**Proof.** Equation (1) implies that
\[ a_n(\mu) = \int d\mu(s) s p_n(\mu; s)^2. \] (13)
The balance relation (5), written in terms of the measure $\sigma$ provides:
\[ a_n(\mu) = \int d\sigma(\beta) \int d\mu(s) \phi_{\beta}(s) [p_n(\mu; \phi_{\beta}(s))]^2 = \] (14)

\[ = \int \int d\sigma(\beta) d\mu(s) \sum_{k,r, k', r'} \Gamma_{k,r}^n \Gamma_{k', r'}^{n-1} p_k(\mu; s) p_{r'}(\sigma; \beta) \times \]

\[ \times [\delta B_k(\mu; s) p_r(\sigma; \beta) + \tilde{\delta} B_r(\sigma; \beta) p_k(\mu; s)], \]

where we have employed eqs. (10), (11). We can compute the integrations explicitly thanks to the orthogonality of the polynomials involved: after some manipulations, one gets
\[ a_n(\mu) = \sum_{k,r} \Gamma_{k,r}^n \left( \delta [a_k(\mu) \Gamma_{k,r}^n + 2b_k(\mu) \Gamma_{k-1,r}^n] + \tilde{\delta} [a_r(\sigma) \Gamma_{k,r}^n + 2b_r(\sigma) \Gamma_{k-1,r}^n] \right). \] (15)
Notice that the r.h.s. is a linear combination of the entries of the Jacobi matrices $J_\mu$ and $J_\sigma$ and that the highest index for both $k$ and $r$ is $n$. Therefore, the term $a_n(\sigma)$ can be easily determined: its linear coefficient is $\delta(\Sigma_{0,n}^\mu)^2$ and this is larger than zero whenever $b_n(\sigma) > 0$ (see eq. 16 below).

**LEMMA 2.** The matrix $\Gamma^{n+1}$ can be computed from $\Gamma^n$, $\Gamma^{n-1}$ and the Jacobi matrix entries $\{a_0(\mu), \ldots, a_n(\mu)\}$, $\{b_1(\mu), \ldots, b_{n+1}(\mu)\}$, $\{a_0(\sigma), \ldots, a_n(\sigma)\}$ and $\{b_1(\sigma), \ldots, b_{n+1}(\sigma)\}$. In addition, the term $b_{n+1}(\sigma)$ only affects the computation of $\Gamma^\sigma_{0,n+1} = \delta\Gamma^\sigma_{0,n} b_{n+1}(\sigma)/b_{n+1}(\mu)$. (16)

**Proof.** We start from the relation

$$b_{n+1}(\mu) p_{n+1}(\mu, s) = (s - a_n(\mu)) p_n(\mu, s) - b_n(\mu) p_{n-1}(\mu, s),$$

and map $s$ in $\phi_\mu(s) = s + \delta \beta$: define $P_{n+1}(s) := b_{n+1}(\mu) p_{n+1}(\mu, \phi_\mu(s))$:

$$P_{n+1}(s) = (\phi_\mu(s) - a_n(\mu)) p_n(\mu, \phi_\mu(s)) - b_n(\mu) p_{n-1}(\mu, \phi_\mu(s)).$$

Use now eqs. (10) and (11) to express the terms at r.h.s. via the matrices $\Gamma^n$ and $\Gamma^{n-1}$. Then, we multiply both sides of eq. (18) by $p_j(\mu, \sigma) \delta^{(j)}$ and integrate w.r.t. $d\sigma d\beta d\mu(s)$, to get

$$b_{n+1}(\mu) \Gamma_{j,l}^{n+1} = \delta (a_j(\mu) \Gamma_{j,j}^{n} + b_{j+1}(\mu) \Gamma_{j,j+1}^{n} + b_j(\mu) \Gamma_{j,j-1}^{n}) + \delta (a_l(\sigma) \Gamma_{j,l}^{n} + b_{l+1}(\sigma) \Gamma_{j,l+1}^{n} + b_l(\sigma) \Gamma_{j,l-1}^{n}) - a_n(\mu) \Gamma_{j,j}^{n} - b_n(\mu) \Gamma_{j,j}^{n-1}.$$

The r.h.s. of these equations is a linear combination of the matrix elements of $\Gamma^n$ and $\Gamma^{n-1}$. The coefficients are given by the matrix entries of $J_\mu$ and $J_\sigma$. Direct inspection, using the triangular nature of the matrices $\Gamma^n$ and $\Gamma^{n-1}$, reveals that the Jacobi entries of highest index appearing are $a_n(\mu)$, $b_{n+1}(\mu)$ and $a_n(\sigma)$, $b_{n+1}(\sigma)$, and that eq. (16) holds.

Remark that in the previous Lemma the quantity $b_{n+1}(\mu)$ is implicitly assumed to be non-null, for otherwise the measure $\mu$ would consist of a finite number of atoms. Finally, we have

**LEMMA 3.** The matrix elements $b_{n+1}(\sigma)$ and $\Gamma_{0,n+1}^{n+1}$ can be computed in terms of $\Gamma_{k,r}^{n+1}$ (with $(k,r) \neq (0,n+1)$) and of $b_{n+1}(\mu)$, provided $\delta < \delta_{n+1}$, so that a solution of the $n+1$ truncated inverse problem exists.

**Proof.** Use the balance relation (5) in $\int d\mu(s) p_{n+1}(\mu, s)^2 = 1$, the normalization relation. Working out terms one gets

$$1 = \sum_{k,r} (\Gamma_{k,r}^{n+1})^2 = \sum_{(k,r) \neq (0,n+1)} (\Gamma_{k,r}^{n+1})^2 + (\delta\Gamma_{0,n}^{n+1} b_{n+1}(\sigma))^2.$$ (19)
where the last term of the summation has been singled out using eq. (16), Lemma 2. Equation (19) can then be solved to find uniquely \(b_{n+1}(\sigma) \geq 0\), provided the difference between one and the first summation at r.h.s. (a positive quantity) is non-negative: this is a consistency condition that tells us when a solution of the \(n+1\)-truncated inverse problem exists. Notice that this condition can be computed much more easily than the determinantal inequalities in (H-M, 1990).

In conclusion, the algorithm consists of the concatenation of these three Lemmas. The sequence of quantities being produced is therefore:

\[
[a_{n-1}(\sigma), b_n(\sigma), \Gamma] \Rightarrow a_n(\sigma) \Rightarrow \Gamma_{r,s}^{n+1} (r,s) \neq (0,n+1) \Rightarrow b_{n+1}(\sigma), \Gamma_{0,n+1}^{n+1}.
\]

5. Exploring the algorithm: Numerical Stability and Approximation

The first mandatory test of the algorithm presented in the previous section is a check of its numerical stability: I have already remarked that many algorithms dealing with these sorts of problems have been plagued by exponentially growing computational errors. After the Jacobi matrix \(J_\sigma\) has been found, we can compute its spectral decomposition in a stable fashion, so obtaining the parameters of the I.F.S.: the fixed points \(\{\beta_j\}_{j=1}^N\) and the weights \(\{\pi_j\}_{j=1}^N\).

We can then use the algorithm of (Man, 1996) or of (Man, 2000) as well to recompute the Jacobi matrix \(J_\mu\) and compare the result with the original matrix, input of the first algorithm. As a test case, we take the Jacobi matrix of the (normalised) Chebyshev polynomials, corresponding to the measure \(d\mu(s) = ds/\pi\sqrt{1-s^2}\): in Fig. 1 errors in the reconstruction are plotted versus \(n\) in doubly logarithmic scale, showing a very mild polynomial increase.

We can also compare the approximation of the sampled measure effected by I.F.S. and by Gaussian integration, eq. (3): in Fig. 2 we plot the distribution functions \(F(x) = \int x^d \mu(s)\) of these measures and of the original Chebyshev measure. Here, we have chosen a very low value \(M = N = 3\): the I.F.S. approximation outperforms Gaussian integration, in particular at the ends of the support and near zero.

Next, we perform the same analysis for the third convolution power of the ternary Cantor measure: this is an example of a fractal measure supported on an interval, \([0,3]\). It can also be seen as the invariant measure of a homogeneous IFS with four (overlapping) maps and \(\delta = 1/3\). Of course, if we set \(N = M = 4\) and the correct value of \(\delta\), the algorithm retrieves the parameters \(\{\pi_j, \beta_j\}_{j=1}^4\) exactly. To explore the approximation power of the technique, we have chosen \(M = 6\) and varied the parameter \(\delta\) in order to reproduce exactly one additional term in the Jacobi matrix. The distribution functions of the original and reconstructed IFS measures are compared in Fig. 3: only small
Figure 1. Absolute differences $\varepsilon_n = |a_n(\mu) - a_n(\mu')| + |b_n(\mu) - b_n(\mu')|$ between $\mu$ and the recomputed $\mu'$, versus $n$. Curves are drawn for five different values of $\delta$ and $N = 80$. $\mu$ is the Chebyshev measure. For $n > N$ differences are obviously of the order of unity.

Figure 2. Distribution functions of the Chebyshev measure (smooth red curve), of the discrete, three points Gaussian approximation (green step function) and of the I.F.S. approximation with $M = 3$ (blue fractal curve).

differences are perceivable. More refined comparisons tests will be described elsewhere.

6. I.F.S. Integration Turning into Gauss

The algorithm we have developed is ideally suited for enjoying the display of the behaviour of $\delta$-H.I.F.S. parameters in various examples. Let us analyze again the classical Chebyshev measure, to show what happens at fixed $N$
when $\delta$ explores the range $[0,\delta_N)$ assured by Thm 2. Remark that the algorithm of section 4 can be used to find the value $\delta_N$: this latter is plotted versus $N$ in Fig. 4, displaying an asymptotic power-law decrease.

For $\delta = 0$ we have $J_\sigma = J_\mu$ and the algorithm of section 4 becomes the identity (although not exactly so, in floating point arithmetics). In the limit of vanishing $\delta$, the fixed points $\beta_j$ tend to the Gaussian points $x_j$ and the map probabilities $\pi_j$ to the weights $w_j$. Figure 5 plots the locations $\beta_j$ versus $\delta$ (green line) together with the first image of the interval $[\beta_1, \beta_3]$ (that is, the convex hull of the support of the reconstructed measure) under the maps $\phi_\beta$. This plot displays a sort of “Gauss tongues” that shrink towards the Gaussian

\[ F(x) \]

\[ x \]

\[ F(x) \]

\[ x \]

\[ F(x) \]

\[ x \]
Figure 5. Fixed points \( \{ \beta_j \}_{j=1}^5 \) versus \( \delta \) (green lines) for the Chebyshev measure, \( M = 5 \) and their Gauss tongues (red stripes). See text for definitions and details.

Figure 6. Fixed points and associated probabilities \( \{ (\beta_j, \pi_j) \}_{j=1}^5 \) for \( M = 5 \), versus \( \delta \) in the case of the uniform Lebesgue measure.

points as \( \delta \) goes to zero. It should be noticed that the plot of these tongues—when not interrupted at the first iteration—is fractal.

Finally, the uniform Lebesgue measure on \([0, 1]\) can be exactly reproduced by \( \delta \)-H.I.F.S. with any number of maps \( M \) with equal weights, equally spaced \( \{ \beta_j \}_{j=1}^M \) and \( \delta = 1/M \). In Fig. 6 we let \( M = 5 \) and we plot the nonlinear behaviour (versus \( \delta \)) of the fixed points and probabilities of the I.F.S. maps that join smoothly the Gaussian points and weights obtained for \( \delta = 0 \) to the exact I.F.S. parameters obtained for \( \delta = 1/5 \).
7. Conclusions

We have presented in this paper a new algorithm for the approximation of a target measure by measures in the class of Iterated Function Systems. The algorithm appears to be numerically stable when implemented in floating point. It can also be coded symbolically, if square roots are avoided in a trivally modified version that relies on monic, rather than normalized, orthogonal polynomials. The algorithm requires the Jacobi matrix of the sampled measure as input: since this is usually needed for conventional Gaussian integration, the new approximation can be performed at no additional sampling cost.

Integration of functions w.r.t. to I.F.S. measures can be efficiently performed and a theory of Richardson extrapolation developed (Dem, 1991). Therefore, also in the light of previous results (B-D, 1991) indicating a definite advantage of I.F.S. above conventional Gaussian integration, we believe that the combination of these techniques, extended for the first time to large orders thanks to our algorithm, might become interesting in various applications, including the study of the Fourier transforms of singular measures and of their orthogonal polynomials (Man, 2006).

Finally, the reversible nature of the transformation between $J_{\sigma}$ and $J_{\mu}$ prompts for a new algorithm for the computation of the Jacobi matrix associated with homogeneous I.F.S. even with uncountably many affine maps, that will be described elsewhere.

References


