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Polygonal billiards: spectrum and transport

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Collaborators

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Chaotic hyperbolic billiard

Chaotic elliptic billiard
In polygonal billiards there is no exponential instability, and even investigation of basic ergodic properties is difficult (and profound)

Gutkin ’86, ’96, ’03, ’12
J. Smillie: “Finally the fact that rational billiards are more complicated than “integrable” systems and yet not fully “chaotic” has led physicists to consider them as test cases for questions relating quantum dynamics to classical dynamics.”

C.P. Dettmann, E.D.G. Cohen: “In addition, the precise role played by microscopic chaos — as represented by the Lyapunov exponents — and “macroscopic chaos”, as embodied by the randomly placed scatterers for the existence of a diffusion process and the value of the diffusion coefficient, remains open. A similar but more complicated situation obtains when diffusion of momentum (viscosity) or energy (heat conduction) and other transport processes are considered.”
Rational vs Irrational

If angles are not rationally connected to π, few rigorous results, but for a highly counteintuitive theorem, (Vorobets, ’97)

For rational triangles ergodicity is ruled out by the fact that one trajectory has a finite set of outgoing angles: the dynamics is foliated into a set of “directional dynamics”

The theorem! Directional dynamics is ergodic for almost all initial angles θ, but never mixing! (Kerckhoff, Masur and Smillie ’86)
Square annulus

Foliation into directional dynamics, associated to the initial angle, $\phi_0$

$\xi$ is the sweeping angle, through which transport is studied

RA, Guarneri, Rebuzzi, ’00; Rebuzzi, RA, ’11 and in preparation
Different classes, according to whether $L_1/L_2$ and $\tan(\phi_0)$ are rational or not: we will just consider the irrational case.

Let $T$ denote the discrete Birkhoff dynamics: globally it cannot be ergodic, since once the initial outgoing angle $\phi_0$ is selected, only a few (3) other outgoing angles can be generated along a trajectory: this leads to introducing a foliation $T_{\phi}$ and this will be the dynamics whose ergodic properties are investigated.

Directional dynamics is almost always (w.r.t. the initial angle) ergodic and never mixing.

This might anticipate a weak mixing as a maximal ergodic property, and non-trivial spectral features of the Koopman operator.
Spectral ergodic theory

Koopman operator on square integrable functions

\[(Uf)(x) = f(Tx)\]

Ergodicity means that the only proper eigenvalue is 1.

If, in the complement of constant functions, the spectrum is absolutely continuous, the system is mixing.

\[C_f(n) = \int d\Omega(z) f(T^nz)f(z) = \int_0^{2\pi} d\mu_f(\omega)e^{i\omega n}\]
Directional dynamics is not mixing -> no correlation decay: but might be weakly mixing:

\[
\ln(\Sigma IC(t)^2 / t) \text{ vs } t
\]

Decay of integrated correlations ruled out by a fractal exponent of the spectral measure: \(D_2\)

Ketzmerick, Petschel & Geisel '92, Holschneider '94, RA, Guarneri, Rebuzzini '00

\(D_2\) from fractal analysis of the spectral measure
Weak mixing: only integrated correlations decay

\[ C_{int}(t) = \frac{1}{t} \int_0^t d\tau |C(\tau)|^2 \]
Figure 2. (a) Phase-averaged (upper line) and integrated (lower) correlation functions of the angle $\xi$, spanned by the radius vector, as a function of time $t$ (measured in numbers of collisions).

The billiard table belongs to class 3 with parameters $l_1/l_2 = \pi/2$ and $\tan \phi = \pi/4$. The dotted line, which fits $C_{int}$, has a slope equal to $-D_2$.

(b) Estimates of generalized dimensions $D_1$ (empty symbols) and $D_2$ (full symbols) of the spectral measure, associated with the angle $\xi$, for the same parameters. The dimension estimates are given by the slopes of straight lines: $D_1 = 0.69 \pm 0.01$ and $D_2 = 0.42 \pm 0.02$. The details are explained in section 6.

Higher order statistics

3.1. Dynamical quantities and spectral analysis

The dynamics inside the billiard table in figure 1 is equivalent to the dynamics of the particles in a two-dimensional infinite periodic lattice with square obstacles, i.e., an generalized Lorentz gas, recently examined in [12–14, 16, 28]. The unfolded system is obtained by reflecting the elementary cell and the segment of a trajectory at each collision point with the external square. Instead of considering particle diffusion along the channels of the extended system, we examine the transport generated by billiard trajectories revolving around the inner square obstacle.

For this purpose, the natural observable is the angle $\xi(z)$, spanned by the radius vector, joining the center of the billiard $O$ with the collision point $z$, when the particle is moving from $z$ to $B\phi z$; it is assumed positive when counter-clockwise. The total angle accumulated by a single particle up to time $t$ is $X_{int}(z, t) = t - \sum_{s=0}^{t-1} \xi(Bs\phi z)$; (6)

/ $X_{int}(z, t)/ (2\pi)$ gives the number of revolutions completed by a trajectory up to the time $t$.

Correlation function

Integrated correlation function
**Other models of transport**

Alonso, Ruiz, Vega, ’02

![Polygonal chain](image)

Jepps, Rondoní, ’06

![Diagram](image)

Sanders, Larralde, ’06

![Diagram](image)
The second moment

RA, Guarneri & Rebuzzini ’00, Rebuzzini, RA ’11

Diffusing variable

\[ \Theta(z, t) = \sum_{s=0}^{t-1} \xi(T^s_{\theta} z) \]

\[ \sigma^2(t) = \int_{\mathcal{M}_\theta} d\Omega(z) \Theta^2(z, t) = \sum_{r,s=0}^{t} C_\xi(r - s) \]

and, in terms of the spectral measure

\[ \sigma^2(t) = \int_{-\pi}^{\pi} d\mu_\xi(\theta) \frac{\sin^2(\theta t/2)}{\sin^2(\theta/2)} \]

which leads to the estimate in terms of \( \alpha \), scaling index at 0 of the spectral measure

\[ \sigma^2(t) \sim t^{2-\alpha} \]
It is easy to see that the inner sum behaves like a real number converges for $\phi$.

In particular, the definition of the dynamical and spectral exponents.

We define the exponent of algebraic growth of the spectral measure at 0.

Orthogonal to constant functions. We note, however, that the measure scales like a constant function.

We have thus found that the exponent is such that the sum converges if $\phi$.

We find that the observable is such that the integral tends to a constant value.

The number is well defined even in the limit of $\phi$.

The number $\phi$ is a ballistic bound.

FIG. 6. The number $\phi$ has given a ballistic bound $\phi$.

In the limit, e.g., of a pure point spectrum clustering at 0.

Continuous component, because such a scaling at 0 may re-appear.

The number $\phi$ is a ballistic bound $\phi$.

To illustrate these general aspects, consider two different classes:

- Class A
- Class C

In case of a class A case, the invariant manifold is the full phase space $\phi$.

In case of a class C case, the squares by circles with the same center $\phi$.

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Strong/weak anomalous diffusion

Full spectrum of transport exponents

\[ \langle |\Theta(t) - \Theta(0)|^q \rangle \sim t^{\nu(q)} \]

normal \( \nu(q) = q/2 \)

"Strong" anomalous diffusion: not a single scaling exponent
Evidence for a single scale for the moments

Figure 5. Scaling function $\gamma(q)$ of the moments $\sigma(q)$, given by (45), plotted versus the order $q$. The dotted straight lines have a slope $q(1-\bar{\alpha}(0)^2)$, according to the theoretical prediction (38), extended to real values of $q$. Moments of integer orders are marked by halos. (a) Same data of previous figures, for which the scaling exponent $\bar{\alpha}(0) = 0.3$ is derived in figure 4. The parameters of (b) are $l_1/l_2 = (\sqrt{5}+1)/2$, $\tan \phi = \pi/4$ and $\bar{\alpha}(0) = 0.47$.

An approximation of the microcanonical measure inside the phase space $M_\phi$ is obtained as follows. The phase average is evaluated by taking a uniform distribution of the particles inside the accessible region of the billiard, and the sign of the components of the unity velocity $\vec{v}$ is assigned at random. The first collision points with the boundaries of the billiard are taken as initial conditions of the Birkhoff mapping.

Spectral analysis of the signals is reconstructed numerically by employing the fast Fourier transform algorithm (FFT), i.e. a finite discrete Fourier transform for vectors in $C_2^T$. Two different methods have been tested to calculate the coarse-grained approximation of the (one-dimensional) average density power spectrum (18). According to (15), $m_\xi(\theta)$ may be approximated by a direct-FFT of the phase-averaged correlation sequence $C_{ph}\xi(t)$. Alternatively, according to (18) and (28), we calculate the square modulus of the direct-FFT of the signal $\xi(Bt)$ of single trajectories and then make a phase average over the initial conditions. Owing to the finite time interval $-T \leq t \leq T$, we multiply sequences, $C_{ph}\xi(t)$ or $\xi(Bt)$, by a proper windowing function \[ w(t) = 1 - |t|/T, \] to reduce the amount of negative values in the former method, we prefer to use a triangular window function instead of a square windowing function as in the theoretical treatment, because its partial Fourier series is a non-negative function. In both methods, the resolution in frequency space is $\Delta 1/\theta = \pi/T$. For the finest resolution, i.e. $T = 220$, the order of magnitude of negative values, in the first method, is $\lesssim 10^{-4}$. Apart from numerical errors, the two methods give the same results, for a fixed resolution. By comparison of the two methods, we get that less than 1% of the total mass on the torus is affected by numerical errors. In figure 4 (a) a construction of $m_\xi(\theta)$ is shown, calculated with a resolution of $\pi/(2^{14})$.

The first two generalized dimensions $D_1$ (information dimension) and $D_2$ (correlation dimension) [43] are calculated by making a sequence of dyadic partitions of the coarse-grained...
Figure 3. Fourth-order moment of \( \langle \xi_1(z, t) \rangle \) versus time in logarithmic scales, for the same data of figure 4. The slope of the straight line (= 3.4) is given by the theoretical prediction (38) with \( n = 4 \) and \( \bar{\alpha}(0) = 0 \).

The occurrence of anomalous diffusion, previously found for \( \sigma^2(t) \) [24], is confirmed for moments of arbitrary orders; in figure 3 the fourth-order moment is shown, as an example.

The crucial issue is to get an estimate for the exponents of the algebraic growth in time of integer moments.

By a generalization of (3), we introduce a multiple-time phase-averaged correlation function of the observable \( \xi \), which depends on \( (n - 1) \) time intervals:

\[
C_{\text{ph}}(\mathbf{t}) = \int M \phi \, d\Omega_1(z) \xi(z) \left( \prod_{l=0}^{n-1} \xi(B_t l \phi z) \right), \quad \mathbf{t} = (t_1, t_2, \ldots, t_{n-1}) \in \mathbb{Z}^{n-1}.
\]

The \( n \)th moment \( \sigma(n)(t) \) can be expressed as a function of \( C_{\text{ph}}(\mathbf{t}) \):

\[
\sigma(n)(t) = t - 1 \sum_{t_0=0}^{t-1} t_1 - 1 \sum_{t_1=0}^{t-2} \cdots t_{n-1} - 1 \sum_{t_{n-1}=0}^{t-n+1} \int M \phi \, d\Omega_1(z) \xi(B_{t_0} \phi z) \xi(B_{t_1} \phi z) \cdots \xi(B_{t_{n-1}} \phi z) = t - 1 \sum_{t_0=0}^{t-1} t_1 - 1 \sum_{t_1=0}^{t-2} \cdots t_{n-1} - 1 \sum_{t_{n-1}=0}^{t-n+1} C_{\text{ph}}(\mathbf{t} - t_0 \mathbf{I}),
\]

where \( \mathbf{I} \) is the vector with all entries equal to 1; note that the \( n \)th-order moment is related to a \((n - 1)\)-point correlation function.
So..

Polygons enjoy weak ergodic properties yet they exhibit nontrivial transport.

They provide an example of “weakly” anomalous transport.